ANALYTICAL EVALUATION OF ECONOMIC RISK CAPITAL AND DIVERSIFICATION USING LINEAR SPEARMAN COPULAS

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Abstract.

The analytical evaluation of economic risk capital as well as the measurement and allocation of diversification for portfolios of non-normal risks is an open field in risk management research. Based on the method of copulas, we construct a parametric family of multivariate distributions using mixtures of independent conditional distributions. We introduce a new family of trivariate copulas that fulfills the four most desirable properties that a multivariate statistical model should satisfy and thus solves an open problem in Joe(1997). The bivariate margins of this family belong to a simple but sufficiently flexible one-parameter family of bivariate copulas, called linear Spearman copula, which is similar but different to the convex family of Fréchet(1958). Significant examples illustrate the discussed concepts and methods.

Key words: economic risk capital, value-at-risk, conditional value-at-risk, diversification effect, risk allocation, copulas, mixtures

1. Introduction.

The present contribution is devoted to the analytical evaluation of economic risk capital (ERC) for portfolios of asset and/or liability risks as well as to the measurement and allocation of diversification in such portfolios.

Emphasis is put on the multivariate statistical modeling of portfolio risks using a parametric family of multivariate distributions with arbitrary marginals, which satisfy all of the four desirable properties required by Joe(1997):

a) There should exist an interpretation like a mixture or other stochastic representation.
b) The margins, at least the univariate and bivariate ones, should belong to the same parametric family and numerical evaluation should be possible.
c) The bivariate dependence between the margins should be described by a parameter and cover a wide range of dependence.
d) The multivariate distribution and density should preferably have a closed-form representation, at least numerical evaluation should be possible.

In general, these desirable properties cannot be fulfilled simultaneously. For example, multivariate normal distributions satisfy properties a), b) and c) but not d). The method of
copulas satisfies property c) but implies only partial closedness under the taking of margins, and can lead to computational complexity as the dimension increases. In fact, it is an open problem to find parametric families of copulas that satisfy all of the desirable properties. In the present paper such a parametric family is constructed. It is based on the method of mixtures of independent conditional distributions. The important bivariate and trivariate models, together with ERC and diversification illustrations, are studied in detail in Sections 8 to 10, as well as in the Appendix. A mathematical analysis of the higher dimensional models is postponed to later studies. A more detailed outline of the content follows.

Section 2 begins with a survey of some risk measures suitable for the evaluation of economic risk capital (ERC) in portfolios of assets and/or liability risks. From the scientific risk management viewpoint the ubiquitous value-at-risk (VaR) measure, which is in general not subadditive, should be rejected and replaced by coherent risk measures like the conditional value-at-risk (CVaR) measure and other risk measures derived from the distorted expected value principle of Denneberg(1990/94) and Wang(1996a). For concrete illustrations and comparisons, we retain the VaR and CVaR measures. Concerning ERC modeling, we distinguish between insurance economic risk capital (I-ERC), discussed in Section 2.1, and market economic risk capital (M-ERC), discussed in Section 2.2. A basic model, which allocates ERC to an asset liability portfolio model, is presented in Section 2.3. An application to life insurance is summarized in Section 2.4.

An introduction to the measurement and allocation of diversification in portfolios of risks is offered in Sections 3 and 4. The diversification effect is defined as the difference between the sum of the risk measures of stand-alone risks and the risk measure of all risks taken together. Typically, the diversification effect should be non-negative, at least for “positive dependent” risks. A counterexample, for which this is not the case, is the variance risk measure, as shown in Section 3.1. Properties of the diversification effect for the VaR and CVaR measures are discussed in Section 3.2. The interesting but non-trivial risk allocation problem, which consists to apportion the non-negative diversification effect of a portfolio of risks in a fair manner to its components, is studied in Section 4. This difficult problem is of great importance. Indeed, the ERC of a firm consisting of several portfolios (e.g. business units, departments, market units, etc.) must under a non-negative diversification effect be less than the sum of the ERC’s of the individual portfolios. Therefore, the “fair” contribution of each single portfolio to the ERC of a firm must be determined. Among the many possible risk allocation principles, only the covariance principle (Section 4.1), the CVaR allocation principle (Section 4.2) and the Shapley principle (Section 4.3) are retained for comparisons.

To evaluate the ERC of portfolios of risks in practice, it is often judicious to express losses using ratios. Then, it is possible to define the loss ratio of a portfolio as a linear combination of the loss ratios of its components, and to use appropriate multivariate statistical models for concrete ERC calculations. A basic portfolio loss ratio model, which realizes this idea and will be used in numerous illustrations, is introduced in Section 5.

The determination of ERC, diversification effect and risk allocation for the VaR and CVaR measures using a multivariate normal distribution is straightforward. Section 6 illustrates this fact at two important examples. Example 6.1 discusses the diversification between assets and liabilities at a simple portfolio model from life insurance. Example 6.2 presents calculations of solvency margins and diversification for non-life insurance companies, a topic relevant in questions related to mergers and acquisitions.

The core of our scientific contribution, which concerns the evaluation of the aforementioned quantities using multivariate non-normal distributions, begins in Section 7. Based on the method of copulas, summarized in Section 7.1, we recall in Section 7.2 the construction of parametric families of multivariate copulas using mixtures of independent conditional distributions. This method is applied in Section 9 to construct a new family of trivariate copulas that satisfies the four desirable properties a)-d). It is first necessary to focus
on a simple but sufficiently flexible one-parameter family of bivariate copulas, called linear Spearman copula, which is similar but different to the convex family of Fréchet(1958) and is introduced and motivated in Section 8.1. The analytical evaluation of the distribution and stop-loss transform of bivariate sums following a linear Spearman copula, required in ERC calculations, is presented in Section 8.2. The dependence parameter of the linear Spearman copula is Spearman’s grade correlation coefficient. In practice, however, often only Pearson’s linear correlation coefficient is available. Stochastic relationships between these two parameters, which allow parameter estimation from each other, are derived in the Appendix.

Finally, applications to the evaluation of ERC and diversification at several significant situations are illustrated in Section 8.3 (bivariate case) and Section 10 (trivariate case).

2. Basic ERC models.

The determination of capital requirement for actuarial and/or financial risks is usually based on an appropriate risk measure that uses the shape of the profit and loss distribution and especially its right tail. The well established value-at-risk (VaR) capital requirement (quantile reserve defined as percentile of the loss distribution) may fail to be subadditive (and thus stimulate diversification) and does not take into account the severity of an incurred adverse loss event. The encountered deficiencies are captured by the notion of coherent risk measure largely discussed in the recent literature (e.g. Arztnner et al.(1997/99), Arztnner(1999), Wirch(1999), Delbaen(2000), Testuri and Uryasev(2000), Acerbi(2001), Acerbi and Tasche(2001a/b)). In a recent synthesis Wirch and Hardy(1999) consider and compare the following risk measures suitable to define the economic risk capital (ERC) associated to a loss random variable $X$ with distribution function $F_X(x)$:

Value-at-Risk (VaR)

$$ VaR_\alpha[X] = Q_\alpha = \inf \{x | F_X(x) \geq \alpha\} $$

This value is the maximum possible loss, which is not exceeded with the probability $\alpha$ (in practice $\alpha = 95\%$ or $\alpha = 99\%$). VaR is in general not subadditive and not a coherent risk measure.

Conditional Value-at-Risk (CVaR)

$$ CVaR_\alpha[X] = \mathbb{E}[X | X > VaR_\alpha[X]] $$

The conditional expected loss given the loss exceeds its value-at-risk represents the “average of the $100\alpha\%$ worst losses” in a random sample of losses. One has the useful formula

$$ CVaR_\alpha[X] = VaR_\alpha[X] + \frac{1}{\pi}(1-CVaR_\alpha[X]), $$

where $m_x(x) = \mathbb{E}[X - x | X > x]$ is the mean excess function, $\pi_x(x) = S_x(x) \cdot m_x(x)$ is the stop-loss transform, $S_x(x) = 1 - F_X(x)$ is the survival function, and $\pi = 1 - \alpha$ is interpreted as loss probability. Some properties of CVaR, which is a coherent risk measure for continuous distributions, are discussed in several recent papers (e.g. Testuri and Uryasev(2000), Acerbi(2001), Acerbi and Tasche(2001a/b), Hürlimann(2001a)).
Distorted Expected Value (DEV)

\[ DEV[X] = \int_{x}^{\infty} (g[S_x(t)] - 1) dt + \int_{0}^{x} g[S_x(t)] dt \]

This value represents an expectation based on the risk adjusted survival function \( g[S_x(x)] \), where \( g : [0, 1] \rightarrow [0, 1] \) is an increasing (concave) function with \( g(0) = 0, g(1) = 1 \), called (concave) distortion function. For concave \( g \) this coherent risk measure has been introduced by Denneberg(1990/94) and has been used as an inception capital adequacy for an insurance contract (e.g. Wang(1996a), Wang et al.(1997), Hürlimann(1998a)). Let us list some important special cases:

a) \( g(x) = 1_{\{x \leq -\alpha\}} \) defines \( VaR_{\alpha}[X] \) but is not concave and does not define a coherent risk measure.

b) \( g(x) = \min\left\{ \frac{x}{1-\alpha}, 1 \right\} \) defines \( CVaR_{\alpha}[X] \).

c) \( g(x) = x^\rho, 0 < \rho \leq 1 \), defines the PH transform studied by Wang(1995a/c, 1996a/b).

d) \( g(x) = \beta(x; a, b) = \frac{1}{\beta(a, b)} \int_{0}^{x} (1-t)^{-1-b^{-1}} dt \), with \( \beta(a, b) \) the beta function, defines the beta transform. Setting \( b = 1 \) yields the PH transform under c) and \( a = 1 \) yields the dual power transform.

e) \( g(x) = x^\rho(1 - \rho \cdot \ln(x)) \) defines the lookback transform or distorted entropy principle studied in Hürlimann(1997/99/2000).

To fix ideas, the focus in the present study is on economic risk capital (ERC) defined by VaR and CvaR. In general, it is useful to make a distinction between ERC’s for liability and/or asset risks.

2.1. Insurance ERC model.

Typically the future value at time \( T \) of a liability risk is modelled by a non-negative claims random variable \( S_T \), whose certainty equivalent \( P_T = H[S_T] \) at time \( t = 0 \), called risk premium in an insurance context, is obtained from a premium calculation principle. For example, if one applies the standard deviation principle one has \( P_T = H[S_T] = \mu_T + \nu_T \cdot \sigma_T \) with \( \mu_T = E[S_T] \) the expected claims, \( \sigma_T = \sqrt{Var[S_T]} \) the standard deviation of the claims, and \( \nu_T \) the loading factor (or risk-reward ratio in a finance context). The pair \( \{S_T, P_T\} \) is interpreted as an insurance contract, whose loss at time \( T \) can be represented as follows:

\[ L_T^L = S_T - P_T = L_T^D + L_T^G, \]  

with \( L_T^D = S_T - E[S_T] \) the deviation from expected claims, \( L_T^G = -(P_T - E[S_T]) \) the negative of the expected gain from the insurance contract. To be able to cover a possible loss \( L_T > 0 \)
or claims \( S_t > P_t \) with a high probability, an insurance manager borrows at time \( t = 0 \) the amount \( I - ERC_0[L_t^0] \), called insurance economic risk capital. At time \( T \), interest is due at the rate \( i_{RC} \) over the period. To guarantee with certainty the value of the borrowed capital at time \( T \), the amount \( I - ERC_0[L_t^0] \) is invested at the riskless rate \( i_f < i_{RC} \). The value of the insurance economic risk capital at time \( T \) is thus \( I - ERC_T[L_t^0] = I - ERC_0[L_t^0] (1 + i_f - i_{RC}) \).

Applying the VaR measure, the insurance economic risk capital decomposes as follows:

\[
I - ERC_T[L_t^0] = VaR_a[L_t^0] = I - ERC_T^0[L_t^0] + I - ERC_T^0[L_t^0],
\]

with

\[
I - ERC_T^0[L_t^0] = VaR_a[L_t^0] = Q_{S_t}(\alpha) - E[S],
\]

(2.2)

The value \( I - ERC_T^0[L_t^0] \) is interpreted as the insurance economic risk capital required to cover claims \( S_t > E[S_t] \) with protection of the expected gain \( (P_t - E[S_t]) \). Since \( I - ERC_0[L_t^0] \) is obtained from \( I - ERC_T^0[L_t^0] \) by subtracting the expected gain, it suffices to consider only \( I - ERC_T^0[L_t^0] \). Similarly, the value of \( I - ERC_0[L_t^0] \) according to the CVaR measure equals

\[
I - ERC_T[L_t^0] = CVaR_a[L_t^0] = I - ERC_T^0[L_t^0] + I - ERC_T^0[L_t^0],
\]

with

\[
I - ERC_T^0[L_t^0] = CVaR_a[L_t^0] = Q_{S_t}(\alpha) - E[S] + \frac{1}{\alpha} \pi S_t Q_{S_t}(\alpha),
\]

(2.3)

One sees that the value of \( I - ERC_T^0[L_t^0] \) according to CVaR exceeds its value according to VaR by an additional stop-loss dependent term, which represents the capital required to cover the average amount at loss beyond \( VaR_a[L_t^0] \).

In general, an insurer has to manage a whole portfolio of liability risks, which is modelled by a vector of non-negative claims random variables \( X = (X_1,\ldots,X_n) \) with joint multivariate distribution function \( F_X(x_1,\ldots,x_n) \) and marginal distributions \( F_i(x) = \Pr(X_i \leq x), \quad i = 1,\ldots,n \).

For example, the multivariate character may stem from a multinational business activity or be the result of a decomposition of the whole liability into several risk classes. The certainty equivalent at time \( t = 0 \) of the liability portfolio consists of a vector of risk premiums \( P = (P_1,\ldots,P_n) \). An important problem, which is a main subject of the present study, is the evaluation of VaR and CVaR for the aggregate liability risk \( S(X) = X_1 + \ldots + X_n \) (with aggregate certainty equivalent \( P = P_1 + \ldots + P_n \)) and their comparisons with the VaR and CVaR of the stand-alone liability risks of the components \( X_i \) of \( X \).

### 2.2. Market ERC model.

Typically the assets of a firm are invested in \( m \) risky securities on the financial market with accumulated rates of return \( R_i \) over the one-period \([0,T], i = 1,\ldots,m \). The investment strategy consists of a portfolio choice \( w = (w_1,\ldots,w_m) \) with \( \sum_{i=1}^m w_i = 1 \), where \( w_i \) is the proportion of wealth invested in security \( i \). The vector of expected accumulated returns is
denoted by \( \mu = (\mu_1, \ldots, \mu_m) \) with \( \mu_i = E[R_i] \), and the covariance matrix is \( \Sigma = (\sigma_{ij}) \) with \( \sigma_{ij} = \text{Cov}[R_i, R_j] \) for \( i, j = 1, \ldots, m \). The accumulated rate of return of an asset portfolio \( w \) is denoted by \( R_w = \sum_{i=1}^{m} w_i R_i \) with mean \( \mu_w = \mu \cdot w^T \) and variance \( \sigma_{w}^2 = w \cdot \Sigma \cdot w^T \). At time \( T \), the loss per unit of invested capital of the asset portfolio is described by the random variable \( L_w^T = 1 - R_w \). The corresponding market economic risk capitals, also denoted \( M - ERC_T[L_w^M] \), according to VaR and CVaR per unit of invested capital are given by the formulas:

\[
\begin{align*}
\text{VaR}_w[L_w^M] &= 1 - Q_{R_w}(\varepsilon), \\
\text{CVaR}_w[L_w^M] &= \text{VaR}_w[L_w^M] + \frac{1}{\varepsilon} (Q_{R_w}(\varepsilon) - \mu_w + \sigma_{R_w}(\varepsilon)).
\end{align*}
\]

(2.4)

(2.5)

Without a specification of the multivariate distribution of \( R = (R_1, \ldots, R_m) \), the quantities (2.4) and (2.5) cannot be calculated. Though alternative choices are possible, as will be demonstrated in the next Sections, one often assumes a multivariate normal distribution of returns, in which case one obtains the explicit formulas:

\[
\begin{align*}
\text{VaR}_w[L_w^M] &= 1 - \mu_w + \Phi^{-1}(\alpha)\sigma_w, \\
\text{CVaR}_w[L_w^M] &= 1 - \mu_w + \frac{1}{\varepsilon} \phi(\Phi^{-1}(\alpha))\sigma_w,
\end{align*}
\]

(2.6)

(2.7)

where \( \Phi(x) \) denotes the standard normal distribution and \( \phi(x) = \Phi'(x) \).

2.3. Asset and liability ERC model.

Modern asset liability management takes into account both the characteristics of the assets and liabilities to make investment decisions. Early and more recent asset liability portfolio models include Wise(1984a/b), Wilkie(1985), Sharpe and Tint(1990), Elton and Gruber(1992), Leibowitz et al.(1992), Keel and Müller(1995), etc. To be self-contained, let us recall the asset liability portfolio model first considered in Hürlimann(2001b) together with the associated economic risk capitals.

At an initial time \( t = 0 \) the assets are valued according to market prices with an initial value \( A_0 \) and are supposed to be tradable on the financial market. After one period at time \( t = T \) the assets with terminal random value \( A_T \) meet the liabilities or claims with random value \( S_T \) and certainty equivalent \( P_T = H[S_T] = \mu_T + \nu_T \cdot \sigma_T \) as in Section 2.1. The assets are modelled as in Section 2.2. To summarize, the asset and liability portfolio is described by the set of quantities

\[
PF_w = \{A_0, S_T, P_T = H[S_T], \mu_T, \sigma_T, \mu, \Sigma, w\}.
\]

(2.8)

Depending on the portfolio choice \( w \), the initial surplus is \( U_0 = A_0 - H[S_T] \), and after one period the final surplus is \( U_T = A_T \cdot R_w - S_T \). The random loss of the asset liability portfolio over the period is defined as the negative increase in surplus at time \( T \) and equals

\[
L_T = U_0 - U_T = (S_T - H[S_T]) + A_0 \cdot (1 - R_w).
\]

(2.9)
To determine the economic risk capital \( ERC_T \), let us use the normalized loss \( L_w \) of the portfolio \( PF_w \) defined by

\[
L_w = \frac{I_T}{A_0} = (R_T - H[R_T]) + (1 - R_w),
\]

where \( R_T = \frac{S_T}{A_0} \) represents the liability or claims rate per unit of the initial investment amount, and the calculation principle \( H[\cdot] \) has the positive homogeneous property such that \( H[R_T] = \frac{H[S_T]}{A_0} \). Denote by \( \mu_r \) and \( \sigma_r \) the mean and standard deviation of \( R_T \) such that

\[
0 = \frac{\mu_r}{A_0}, \quad \sigma_r = \frac{\sigma_r}{A_0}.
\]

To examine and point out diversification effects between assets and liabilities, it is of interest to look separately at the ERC of the normalized liability or insurance loss \( L^L_w = R_T - H[R_T] \), called insurance economic risk capital and abbreviated I-ERC as in Section 2.1, and at the ERC of the normalized asset or market loss \( L^M_w = 1 - R_w \), called market economic risk capital and abbreviated M-ERC as in Section 2.2. In the simple case that \( (R_T, R_1, ..., R_m) \) follows a multivariate normal distribution and \( \rho \) denotes the correlation coefficient between \( R_T \) and \( R_w \), one obtains the formulas (Propositions 2.1 and 2.2 in H"urlimann(2001b)):

\[
\begin{align*}
\text{ERC}_T[L_w] &= 1 - \mu_w - \nu_T \sigma_r + \alpha^\ast \cdot \sqrt{(\sigma_w - \rho \cdot \sigma_r)^2 + (1 - \rho^2) \sigma_r^2}, \\
1 - \text{ERC}_T[L^L_w] &= (\alpha^\ast - \nu_T) \cdot \sigma_r, \\
M - \text{ERC}_T[L^M_w] &= 1 - \mu_w + \alpha^\ast \cdot \sigma_w,
\end{align*}
\]

where one sets \( \alpha^\ast = \Phi^{-1}(\alpha) \) for the VaR measure respectively \( \alpha^\ast = \frac{1}{1 - \alpha} \phi[\Phi^{-1}(\alpha)] \) for the CVaR measure. The resulting diversification effects are discussed in Section 6.

### 2.4. Economic risk capital for life insurance.

An important and main example to which the considered asset liability portfolio model applies is life insurance, including the special instance of pensions funds. In this situation ERC modelling has been considered in details by Ballmann and H"urlimann(2001). Recall that, refraining from the cost process, the random loss over a period \( [t, t+1) \) of a life business at time \( t+1 \) can be represented as

\[
L_{t+1} = (DK_t + \pi_T) \cdot (i^w_t - I^w_{t+1}) + (S_{t+1} - \pi^R_T(1 + i^w_t)),
\]

where \( DK_t \) are the mathematical reserves at time \( t \), \( \pi_T, \pi^R_T \) are the net and risk premiums due at time \( t \), \( i^w_t \) is the guaranteed technical interest rate, \( S_{t+1} \) are the aggregate claims at time \( t+1 \) (stochastic sum of the difference between individual benefits and mathematical reserves over all claims in the period \( [t, t+1) \)), and \( I^w_{t+1} = I^w_{t+1} \cdot w^T \) is the rate of return over
the period \([t, t+1]\) of an asset portfolio with weights \(w\) and vector of returns \(I_{t+1} = (I_{t+1}^1, \ldots, I_{t+1}^n)\). Comparing (2.12) with (2.9) it appears adequate to define the asset liability portfolio (2.8) as follows:

\[
P_{w}^{PF} = \{DK_t + \pi_t, S_{t+1}, H[S_{t+1}] = \pi^g_t (1 + i^g_t), \mu_{S_{t+1}}, \sigma_{S_{t+1}}, \mu - i^g_t \cdot e, \Sigma, w\},
\]

\[
e = (1, \ldots, 1), \quad \sigma_{\theta} = \text{Cov}[I_{t+1}^1, I_{t+1}^i]
\]

(2.13)

It is assumed that the rate of return \(I_{t+1}^w\) is independent of the aggregate claims \(S_{t+1}\), hence \(\text{Cov}[R_w, R_T] = 0\) and \(\rho_w = 0\). An assumption of normally distributed aggregate claims in life insurance seems justified for practical ERC calculations, as shown in Ballmann and Hürlimann(2001), Section 6. The value-at-risk and expected shortfall economic risk capitals associated to the normalized losses of \(P_{w}^{PF}\) are calculated according to the formulas (2.11). Measured in units of the invested capital \(DK_t + \pi_t\), the economic risk capital of a life insurance business at time \(t+1\) is given by the formulas

\[
\begin{align*}
\text{ERC}_{t+1} &= \alpha^* \sqrt{\sigma_w^2 + \sigma_R^2} - \mu_w - \nu_T \sigma_R + i_0, \\
1 - \text{ERC}_{t+1} &= (\alpha^* - \nu_T \cdot \sigma_R), \\
M - \text{ERC}_{t+1} &= i_0 - \mu_w + \alpha^* \cdot \sigma_w, \\
\end{align*}
\]

(2.14)

where one sets

\[
\begin{align*}
\sigma_R &= \frac{\sigma_{S_{t+1}}}{DK_t + \pi_t}, \\
\nu_T &= \frac{\pi^g_t (1 + i^g_t) - \mu_{S_{t+1}}}{\sigma_{S_{t+1}}}, \\
i_0 &= i^g_t.
\end{align*}
\]

(2.15)

(2.16)

(2.17)

3. **Diversification.**

Though an old idea, the measurement and allocation of diversification in portfolios of asset and/or liability risks is a difficult problem, which has found so far many answers. By definition, the *diversification effect* is the difference between the sum of the risk measures of stand-alone risks or portfolios and the risk measure of all risks or portfolios taken together, which is typically non-negative. The *risk allocation problem* consists to apportion the diversification effect to the risks or portfolios in a fair manner, to obtain new, firm-internal risk measures of the risks or portfolios. Historically, the first mathematical approach to diversification is due to Markowitz(1952/59/87/94), whose classical portfolio selection model applies to the efficient diversification of investments.

We begin with the diversification effect, which is determined for the variance risk measure in Section 3.1 and for the ERC risk measure in Section 3.2. In Section 4 we present and discuss several risk allocation principles. These results are illustrated at some important situations later in Sections 6 to 10. A more advanced discussion of efficient diversification in the mean-ERC model using mean-variance analysis is found in Hürlimann(2001b).
3.1. Diversification for the variance risk measure.

Consider a portfolio \( X = (X_1, \ldots, X_n) \) with joint multivariate distribution \( F_X(x_1, \ldots, x_n) \) and continuous marginals \( F_i(x) = \Pr(X_i \leq x), \ i = 1, \ldots, n \). We assume the existence of the mean vector \( \mu = (\mu_1, \ldots, \mu_n) \), where \( \mu_i \) is the mean of \( X_i \), and the covariance matrix \( \Sigma = (\sigma_{ij}) \), where \( \sigma_{ij} = \rho_{ij} \sigma_i \sigma_j \), with \( \sigma_i^2 = \sigma_i \) the variance of \( X_i \) and \( \rho_{ij} \) the correlation coefficient between \( X_i \) and \( X_j \), \( i, j = 1, \ldots, n \). The overall risk is \( X = X_1 + \ldots + X_n \) with mean \( \mu \) and variance \( \sigma^2 = \sum \rho_{ij} \sigma_i \sigma_j \). One is interested in the risk measures associated to the losses \( L_i = X_i - \mu_i, \ i = 1, \ldots, n \), \( L = X - \mu \), defined as deviations of the risks from their means, the latter quantities being interpreted as benchmarks. This viewpoint applies in particular directly to the decomposition (2.1). A risk measure \( R \) is a function acting on the set of all losses such that \( R[L] \) is the risk measure of the overall loss and \( R[L_i] \) are the risk measures of the stand-alone losses. The diversification effect \( D_R \) with respect to the risk measure \( R \) is the function acting on the set of all multivariate losses defined by

\[
D_R[L_1, \ldots, L_n] = \sum_{i=1}^{n} R[L_i] - R \left[ \sum_{i=1}^{n} L_i \right].
\]

(3.1)

For the determination of ERC it is natural to postulate a non-negative diversification effect, which is equivalent with the subadditive property of the risk measure. Merging risks does not create extra risk. If a firm must meet a requirement of extra economic risk capital that did not satisfy this property, the firm might separate in subunits requiring less capital, a matter of concern for the supervising authority.

In the classical mean-variance model by Markowitz the risk measure is the variance, that is \( R[L] = \text{Var}[L] \) for all loss random variables \( L \). The diversification effect is given by

\[
D_{\text{Var}}[L_1, \ldots, L_n] = \sum_{i=1}^{n} \sigma_i^2 - \sigma^2 = -\sum_{i \neq j} \rho_{ij} \sigma_i \sigma_j.
\]

(3.2)

The diversification effect is non-negative with certainty only if \( \rho_{ij} \leq 0 \) for all \( i \neq j \). However, most multivariate losses are at least partially positively correlated and the diversification effect will often be negative. Therefore the variance risk measure is not appropriate for the determination of the economic risk capital.

3.2. Diversification for the VaR and CVaR measures.

Consider first the value-at-risk approach for which \( R[L] = VaR_\alpha[L] \). Using (2.2) one obtains immediately the diversification effect

\[
D_{VaR_\alpha}[L_1, \ldots, L_n] = \sum_{i=1}^{n} Q_{x_i}(\alpha) - Q_{x}(\alpha).
\]

(3.3)

It is well-known that this may be negative, where an interesting example in the options market context is discussed in Wirch(1999) and Wirch and Hardy(1999). Under a multivariate normal distribution one has
\[ D_{\text{VaR}_\alpha}[L_1, \ldots, L_n] = \Phi^{-1}(\alpha) \left[ \sum_{i=1}^n \sigma_i - \sqrt{\sum_{i,j=1}^n \rho_{ij} \sigma_i \sigma_j} \right], \]  
\[ (3.4) \]

which is always non-negative whatever the \( \rho_{ij} \)'s are. If \( \rho_{ij} \geq 0 \) (case of positive dependence) the non-negative diversification effect satisfies the best bounds

\[ 0 \leq D_{\text{VaR}_\alpha}[L_1, \ldots, L_n] \leq \Phi^{-1}(\alpha) \left[ \sum_{i=1}^n \sigma_i - \sqrt{\sum_{i=1}^n \sigma_i^2} \right], \]  
\[ (3.5) \]

where the lower bound is attained for perfectly correlated risks \( \rho_{ij} = 1 \), and the upper bound holds for uncorrelated risks \( \rho_{ij} = 0 \). In general, it is natural to consider only “positive dependent” multivariate distributions for which a similar equality holds, that is a maximum diversification effect for independent risks and a vanishing diversification effect for perfect dependence. As is well-known, the latter situation occurs when the risks \( X_1, \ldots, X_n \) are mutually comonotonic, and therefore quantities behave additively, that is \( \sum_{i=1}^n Q_{X_i}(\alpha) = Q_X(\alpha) \) (e.g. Landsberger and Meilijs(1994), Denneberg(1994), Kaas et al.(2000)). Examples of this modelling approach are presented in Sections 6, 7 and 8. Similarly, for the conditional value-at-risk measure \( R[L] = CVaR_\alpha[L] \) one obtains

\[ D_{CVaR_\alpha}[L_1, \ldots, L_n] = \sum_{i=1}^n CVaR_\alpha[X_i] - CVaR_\alpha[X]. \]  
\[ (3.6) \]

In contrast to the variance and value-at-risk measure, the diversification effect is always non-negative, whatever the specification of the multivariate distribution of risks, because the conditional value-at-risk measure is a coherent risk measure and therefore subadditive. Again, it is natural to consider only positive dependent multivariate distributions for which there is a maximum diversification effect for independent risks and a vanishing diversification effect for perfect dependence. In the latter situation, that is \( X_1, \ldots, X_n \) are mutually comonotonic, one knows that \( \sum_{i=1}^n CVaR_\alpha[X_i] = CVaR_\alpha[X] \) (e.g. Hürlimann(2001a), Theorem 2.3).

### 4. Risk allocation principles.

Once a non-negative diversification effect has been quantified, there remains the interesting and non-trivial problem of allocating the overall risk measure of a portfolio in a fair manner to its components. This difficult problem is of great importance. Indeed, the ERC of a firm consisting of several portfolios (e.g. business units, departments, market units, etc.) must under a non-negative diversification effect be less than the sum of the ERC’s of the individual portfolios. Therefore, the “fair” contribution of each single portfolio to the ERC of a firm must be determined. In the following several allocation principles are described.

For a portfolio of losses \( L = (L_1, \ldots, L_n) \) with overall loss \( L = L_1 + \ldots + L_n \), single risk measures \( R[L_i] \) and overall risk measure \( R[L] \), the diversification effect \( D_{\text{VaR}_\alpha}[L_1, \ldots, L_n] \) is given by (3.1). The risk contribution of a loss \( L_i \) to the portfolio loss \( L \) is denoted \( R[L_i|L] \)
and produces the individual diversification effect \( D_r[L_i] = R[L_i] - R[L_i|L] \). By definition of a risk allocation principle one has \( \sum_{i=1}^{n} R[L_i|L] = R[L] \), which in particular implies the additive relationship \( \sum_{i=1}^{n} D_r[L_i] = D_r[L_1,...,L_n] \). That is, the sum of the individual diversification effects is equal to the portfolio diversification effect.

4.1. The covariance principle.

Consider a portfolio of losses \( \mathbf{L} = (L_1,...,L_n) = (X_1 - \mu_1,...,X_n - \mu_n) \) and assume the risk measure is the variance. The loss of the portfolio is \( L = L_1 + ... + L_n = X - \mu \), and it has the risk measure \( R[L] = Var[X] \). Requiring an additive allocation, it is natural to define the risk contribution of a component by

\[
R[L_i|L] = Cov[L_i|L] = Cov[X_i,X], \quad i = 1,...,n .
\]  

(4.1)

This can be rewritten as

\[
R[L_i|L] = \frac{Cov[X_i,X]}{Var[X]} \cdot R[L], \quad i = 1,...,n ,
\]  

(4.2)

which expresses the risk contribution of a component in terms of the portfolio risk. The use of the formula (4.2) for an arbitrary risk measure is called covariance principle. This general allocation principle goes back to Borch(1982) and has been justified using at least four different approaches, as shown in Hürlimann(1998b).

4.2. The conditional value-at-risk allocation principle.

It is interesting to derive an allocation principle from the CVaR measure. Since \( R[L] = CVaR_{\alpha}[L] = E[X|X \geq VaR_{\alpha}[X]] - E[X] \) it is natural to define the risk contribution of a loss component by

\[
R[L_i|L] = E[X_i|X \geq VaR_{\alpha}[X]] - E[X_i], \quad i = 1,...,n ,
\]  

(4.3)

which clearly defines an allocation principle because \( \sum_{i=1}^{n} R[L_i|L] = R[L] \). To illustrate, suppose \( X = (X_1,...,X_n) \) follows a multivariate elliptical density function with mean \( \mu \) and positive definite covariance matrix \( C \) given by

\[
f(x) = \frac{1}{\sqrt{\det(C)}} g[(x - \mu)^T \cdot C^{-1} \cdot (x - \mu)],
\]  

(4.4)

where \( g : [0,\infty) \to [0,\infty) \) is some appropriate function (consult Fang, Kotz and Ng(1987) for background on elliptical distributions). It is known that the conditional distribution of \( X_i \) given \( X = X_1 + ... + X_n \) is again elliptical with conditional mean
\[ E[X_i | X] = E[X_i] - \frac{\text{Cov}[X_i, X]}{\text{Var}[X]} (X - E[X]), \quad i = 1, \ldots, n. \quad (4.5) \]

It follows immediately that

\[ R[L_i | L] = \frac{\text{Cov}[X_i, X]}{\text{Var}[X]} \cdot R[L], \quad i = 1, \ldots, n. \quad (4.6) \]

In this situation the conditional value-at-risk allocation principle is equivalent to the covariance principle. This property can be used to show that a portfolio selection model based on the conditional value-at-risk coincides with the classical mean-variance portfolio selection model provided the returns follow a multivariate elliptical distribution (Hürlimann(2001c), Theorem 3.1). The latter result generalizes the corresponding result for a multivariate normal distribution, which has been shown in a less direct way by Rockafellar and Uryasev(2000), Proposition 4.1.

### 4.3. The Shapley principle.

In a way similar to the axiomatic definition of coherent risk measures by Arztner et al.(1999), it is possible to consider the axiomatic definition of coherent risk allocation principles as in Delbaen and Denault(2000).

**Definition 4.1.** Given a portfolio of losses \( L = (L_1, \ldots, L_n) \) with aggregate loss \( L = L_1 + \cdots + L_n \), one says that a risk allocation \( R[L_i | L] \) \( i \in N = \{1, \ldots, n\} \) is coherent if it satisfies the four properties:

1. **(CA1)** \( \sum_{i=1}^n R[L_i | L] = R[L] \) (full allocation)

   The economic risk capital of a portfolio should be completely allocated to its components.

2. **(CA2)** \( \sum_{i \in M} R[L_i | L] \leq R[\sum_{i \in M} L_i] \quad \forall \ M \subseteq N \) (no undercut)

   No risk manager, or coalition of risk managers, can argue that it would be better off on its own than with the firm, and as a consequence request a lower risk allocation. For short, all the participants should benefit from a positive diversification effect. An equivalent definition, which justifies the name of this axiom, is the following one:

   \( (\text{CA2'}) \) There is no \( M \subseteq N \) and \( H_i, i \in M \), such that

   \[ \sum_{i \in M} H_i = R[ \sum_{i \in M} L_i ] \quad \text{and} \quad H_i < R[L_i | L] \quad \forall \ i \in M. \]

3. **(CA2**) There is no \( M \subseteq N \) and \( H_i, i \in M \), such that

   \[ \sum_{i \in M} H_i = R[ \sum_{i \in M} L_i ] \quad \text{and} \quad H_i < R[L_i | L] \quad \forall \ i \in M. \]

   There is no subset \( M \) of the set of portfolios, such that an allocation of the subset’s ERC exists, which is cheaper for every single component in \( M \).
If joining losses $L_r, r \in M \subseteq N - \{i, j\}$, to a single loss $\sum_{r \in M} L_r$, the losses $L_i$ and $L_j$ make the same risk contribution to the overall loss $L^{(M)} = (\sum_{r \in M} L_r) + L_i + L_j + \sum_{x \in N - \{i, j\}} L_x$, that is $R[L_i | L^{(M)}] = R[L_j | L^{(M)}]$, then one has $R[L_i | L] = R[L_j | L]$ (symmetry).

A risk allocation depends only on the risk contributions within the portfolio, and nothing else.

A riskless component with deterministic loss is exactly allocated its loss.

It is often easy to show that a specific allocation principle is not coherent, but it seems quite difficult to prove the existence of coherent allocation principles.

**Example 4.1.**

The naive proportional allocation principle

$$R[L_i | L] = \frac{R[L_i]}{\sum_{j=1}^{n} R[L_j]}, \quad i = 1, \ldots, n.$$  \hspace{1cm} (4.7)

is not coherent.

Delbaen and Denault(2000) apply some important results from game theory to construct coherent allocation principles. In particular the Shapley value in the space of all games with $n$ players (Shapley(1953), Roth(1988)) can be used to construct the Shapley principle, which is algebraically defined by the explicit formula:

$$R[L_i | L] = \sum_{M \in C_i} \frac{1}{m} \cdot \frac{R[\sum_{j \in M} L_j] - R[\sum_{j \in M - \{i\}} L_j]}{\binom{n}{m}},$$  \hspace{1cm} (4.8)

where $C_i$ represents the set of all subsets of $\{1, \ldots, n\}$ that contain $i$ and $m = \text{card}(M)$. The Shapley value can be interpreted as follows. Suppose the players agree to meet in a room at a certain time. Assume they will arrive at slightly different random times and that the orders of arrival are uniformly distributed. Then the Shapley value is the average contribution of a player to the group’s risk measure upon arrival. Note that the computational time required to evaluate (4.8) is exponential in the number $n$ of players because an explicit evaluation of the risk measure for each of the $2^n$ coalitions is required. However, in real-life applications, $n$ could be rather small, say $n = 5$, then computational time is not a critical issue.

There is an important axiomatic characterization of the Shapley value, which under certain circumstances leads automatically to a coherent allocation principle. Considering values of games, which are allocation principles in the game-theoretic sense, it is known that the Shapley value is the only value that leads to an (additive) allocation principle satisfying the
coherence axioms but for the “no undercut” axiom (CA2) (Shapley(1953), Delbaen and Denault(2000), Definition 8). Note, however, that the additive property in bracket means additivity over games, which is not required for coherent allocation principles. The construction of a unique coherent allocation principle remains thus an open problem. When does the Shapley value satisfy the “no undercut” property, yielding automatically a coherent allocation principle? Delbaen and Denault(2000), Theorem 6, show that this holds under the following rather strong sufficient condition. If for all coalitions \( M, \quad \text{card}(M) \geq 2 \),

\[
\sum_{T \subseteq M} (-1)^{\text{card}(M) - \text{card}(T)} R \left[ \sum_{i \in T} L_i \right] \leq 0 ,
\]

then the Shapley principle is a coherent allocation principle.

5. **A basic portfolio loss ratio model.**

To evaluate the economic risk capital of portfolios of risks in practice, it is often judicious to express losses using ratios. For insurance risks loss ratios are quotients from claims to risk premiums while for financial risks loss ratios are just negative returns. It is then possible to define the loss ratio of a portfolio as a linear combination of the loss ratios of its components. To fix ideas let us work with portfolios of insurance risks.

Consider a portfolio of risks with vector of random claims \( S=(S_1,\ldots,S_n) \), with deterministic vector of risk premiums \( P=(P_1,\ldots,P_n) \), and with vector of random loss ratios \( X=(X_1,\ldots,X_n) \), where \( X_i = \frac{S_i}{P_i}, \quad i=1,\ldots,n \). Let \( P=P_1+\ldots+P_n \) be the aggregate risk premium of the portfolio, and let \( w=(w_1,\ldots,w_n) \) with \( w_i = \frac{P_i}{P} \) the weight associated to risk \( i, i=1,\ldots,n \). Then the loss ratio \( X \) of the portfolio is defined by the linear combination

\[
X = \sum_{i=1}^{n} w_i X_i .
\]

We assume the existence of the mean vector \( \mu=(\mu_1,\ldots,\mu_n) \), where \( \mu_i = E[X_i] \) is the mean, and the covariance matrix \( \Sigma = (\sigma_{ij}) \) such that \( \sigma_i^2 = Var[X_i] \) is the variance and \( \rho_{ij} \) is the correlation between the risks \( X_i \) and \( X_j, \quad i, j=1,\ldots,n \). Moreover, a joint multivariate distribution function \( F_X(x_1,\ldots,x_n) \) with marginal distributions \( F_i(x), \quad i=1,\ldots,n \), has to be specified, which is compatible with the parameters \( \mu, \Sigma \). In terms of the defined loss ratios, one is interested in the stand-alone losses \( L_i = P_i \cdot (X_i - \mu_i), \quad i=1,\ldots,n \), and in the portfolio loss \( L = P \cdot (X - \mu) \). Using (5.1) the mean and variance of the portfolio loss ratio are

\[
\mu_w = \mu \cdot w^T , \quad \sigma^2_w = w \cdot \Sigma \cdot w^T .
\]

If the risk measure is positive homogeneous, an axiom satisfied by a coherent risk measure, that is such that \( R[\alpha L] = \alpha R[L] \) for all \( \alpha > 0 \) and all losses \( L \), then the risk measures associated to the portfolio \( L=(L_1,\ldots,L_n) \) are obtained from the risk measures of the loss
ratios through simple multiplication. For example, if one wants to calculate ERC’s using the value-at-risk and the conditional value-at-risk method, one obtains from (2.2) and (2.3):

$$\text{VaR}_\alpha[L] = (Q_X(\alpha) - \mu) \cdot P, \quad \text{CVaR}_\alpha[L] = \left( Q_X(\alpha) - \mu + \frac{1}{\epsilon} \pi_X(Q_X(\alpha)) \right) \cdot P,$$

$$\text{VaR}_\alpha[L_i] = (Q_X(\alpha) - \mu_i) \cdot P_i, \quad i = 1, \ldots, n,$$

$$\text{CVaR}_\alpha[L_i] = \left( Q_X(\alpha) - \mu_i + \frac{1}{\epsilon} \pi_X(Q_X(\alpha)) \right) \cdot P_i, \quad i = 1, \ldots, n.$$ (5.3)

It is then possible to calculate the diversification effect and consider risk allocations as presented in Sections 3 and 4.

For practical evaluation, there remains the modeling choice for the joint multivariate distribution function $F_X(x_1, \ldots, x_n)$, which will be discussed in Sections 6 to 10, and the statistical estimation of the parameters $\mu, \Sigma$. Provided historical data is available, the latter task is solved as follows.

Let $S^j = (S_1^j, \ldots, S_n^j)$ and $P^j = (P_1^j, \ldots, P_n^j)$, $j = 1, \ldots, m$, represent the claims and risk premiums of the portfolio in $m$ past periods. Consider the loss ratios $X_i^j = \frac{S_i^j}{P_i^j}$, and the weights $w_i^j = \frac{P_i^j}{P_i^*}$, $P_i^* = \sum_{j=1}^m P_i^j$, $i = 1, \ldots, n$, $j = 1, \ldots, m$. Interpreting the historical loss ratios $X_i^j$ as outcomes of the loss random variables $X_i$ with probability function $\text{Pr}(X_i = X_i^j) = w_i^j$, one obtains the mean and variance parameters

$$\mu_i = E[X_i] = \sum_{j=1}^m w_i^j X_i^j,$$

$$\sigma_i^2 = \text{Var}[X_i] = \sum_{j=1}^m w_i^j \cdot (X_i^j - \mu_i)^2, \quad i = 1, \ldots, n. \quad (5.4)$$

To estimate the correlation coefficients $\rho_{ij}$, $i \neq j$, consider the sub-portfolios covering the risks with loss ratios $X_i$ and $X_j$, whose overall loss ratios are determined by the weighted mean

$$X_{ij} = \frac{P_i^*}{P_i^* + P_j^*} X_i + \frac{P_j^*}{P_i^* + P_j^*} X_j, \quad i \neq j, i, j = 1, \ldots, n. \quad (5.5)$$

The associated historical data consists of loss ratios $X_{ij}^k$ and weights $w_{ij}^k$ defined by

$$X_{ij}^k = \frac{P_i^k}{P_i^k + P_j^k} X_i^k + \frac{P_j^k}{P_i^k + P_j^k} X_j^k,$$

$$w_{ij}^k = \frac{P_i^k + P_j^k}{P_i^* + P_j^*}, \quad i \neq j, i, j = 1, \ldots, n, k = 1, \ldots, m. \quad (5.6)$$
Interpreting the $X^k_{ij}$'s as outcomes of $X_{ij}$ with probability function $Pr(X_{ij} = X^k_{ij}) = w^k_{ij}$, one obtains the mean and variance parameters

$$
\mu_{ij} = E[X_{ij}] = \sum_{k=1}^{m} w^k_{ij} X^k_{ij} = \frac{P^*_i}{P^*_i + P^*_j} \mu_i + \frac{P^*_j}{P^*_i + P^*_j} \mu_j,
$$

(5.7)

$$
\sigma^2_{ij} = Var[X_{ij}] = \sum_{k=1}^{m} w^k_{ij} (X^k_{ij} - \mu_{ij})^2.
$$

(5.8)

Note that the formula (5.7) is compatible with the definitions (5.4) and (5.5). In order that (5.8) is also compatible with (5.4) and (5.5), the correlation coefficient $\rho_{ij}$ must satisfy the relationship

$$
\rho_{ij} = \frac{1}{2} \left( \frac{(P^*_i + P^*_j)\sigma_{ij}^2 - [P^*_i \sigma_i^2] - [P^*_j \sigma_j^2]}{(P^*_i \sigma_i)(P^*_j \sigma_j)} \right).
$$

(5.9)

Using (5.4) and (5.9) the mean and variance of the portfolio loss ratio are given by (5.2).

6. Multivariate normal ERC models.

The determination of ERC’s using a multivariate normal distribution is straightforward. It has been illustrated for an asset and liability portfolio model in Section 2.3, with an application to life insurance in Section 2.4. The calculation of ERC’s for the basic portfolio loss ratio model of Section 5 under a joint multivariate normal distribution of the loss ratios is similarly simple. Using (5.3) one finds

$$
VaR_\alpha[L] = \Phi^{-1}(\alpha) \sigma \cdot P, \quad CVaR_\alpha[L] = \frac{1}{1-\alpha} \varphi[\Phi^{-1}(\alpha)] \sigma \cdot P, \\
VaR_\alpha[L_i] = \Phi^{-1}(\alpha) \sigma_i \cdot P_i, \quad CVaR_\alpha[L_i] = \frac{1}{1-\alpha} \varphi[\Phi^{-1}(\alpha)] \sigma_i \cdot P_i, \quad i = 1, \ldots, n.
$$

(6.1)

It is immediate but instructive to illustrate diversification effects and risk allocations at some normally distributed portfolio models.

Example 6.1: Diversification between assets and liabilities

For the asset liability portfolio model of Section 2.3 the ERC formulas (2.11) reveal the subadditive property

$$
ERC_T[L_w] \leq I - ERC_T[L^L_w] + M - ERC_T[L^M_w],
$$

(6.2)

which follows from the inequality $(\sigma_w - \rho_w \sigma_u)^2 + (1 - \rho_w^2)\sigma_u^2 \leq (\sigma_w + \sigma_u)^2$ always valid because $\rho_w \geq -1$. Performing asset and liability management simultaneously has a diversification advantage compared to a separate asset management and liability management. If strict inequality holds in (6.2), then a participant of a global asset liability market can sell $I - ERC_T[L^L_w]$ and $M - ERC_T[L^M_w]$ separately and buy back $ERC_T[L_w]$, making a riskless
profit of amount \( I - ERC_T[L'_w] + M - ERC_T[L'_u] - ERC_T[L_w] > 0 \). To avoid such arbitrage opportunities, one applies a risk allocation principle from Section 4 such that after diversification one has \( ERC_T[L'_w] = I - ERC_T[L'_u|L'_w] + M - ERC_T[L'_u|L_w] \). Applying the simple covariance principle one obtains

\[
I - ERC_T[L'_w] = E[L'_w] + \frac{\text{Cov}[L'_u, L'_w]}{\text{Var}[L'_w]} \cdot (ERC_T[L'_w] - E[L'_w]), \tag{6.3}
\]

\[
M - ERC_T[L'_u|L'_w] = E[L'_u|L'_w] + \frac{\text{Cov}[L'_u, L'_w]}{\text{Var}[L'_w]} \cdot (ERC_T[L'_u|L'_w] - E[L'_u|L'_w]). \tag{6.4}
\]

Let us illustrate at a simple asset liability portfolio model from life insurance as in Section 2.4. There are three asset categories consisting of a riskless asset with accumulated return \( \mu_1 = 0.03 \) and volatility \( \sigma_1 = 0 \), a bond portfolio with return \( \mu_2 = 0.05 \) and volatility \( \sigma_2 = 0.05 \), and an equity portfolio with return \( \mu_3 = 0.10 \) and volatility \( \sigma_3 = 0.15 \). Let \( \rho_{23} = 0.5 \) be the correlation between the bond and equity portfolios. The covariance matrix \( \Sigma = (\rho_{ij} \sigma_i \sigma_j) \) is given by

\[
\Sigma = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0.0025 & 0.00375 \\
0 & 0.00375 & 0.0225
\end{pmatrix}. \tag{6.5}
\]

Suppose the life insurance business invests the amount \( DK_0 + \pi = 50'000 \text{ Mio.} \), and the net risk premium at the end of the period reads \( \pi(1+i^T_t) = 500 \text{ Mio.} \), where \( i^T_t = 0.035 \) is the technical interest rate. The aggregate claims have a mean \( \mu_{S,t} = 375 \text{ Mio.} \) and a standard deviation \( \sigma_{S,t} = 96.25 \text{ Mio.} \). According to (2.15)-(2.17) the liability characteristics are herewith

\[
\sigma_g = 0.00193, \quad \nu_t = 1.55844, \quad i_0 = 0.035. \tag{6.6}
\]

We distinguish between two ERC variants. The first variant is identical to the model in Section 2.4. In the second variant we suppose that the insurance ERC should protect the expected insurance gain, and thus identify with the notion \( I - ERC^D \) in Section 2.1. In this situation the formulas for I-ERC and M-ERC in (2.14) are replaced by

\[
I - ERC_{t+1} = \alpha^* \cdot \sigma_g,
\]

\[
M - ERC_{t+1} = i_0 - \mu_u + \alpha^* \cdot \sigma_u - \nu_T \sigma_R. \tag{6.7}
\]

Risk allocation is done using the covariance principle (6.3), (6.4) noting that in the second variant the losses \( L'_u, L'_u \) must be modified adding respectively subtracting the expected gain \( -\nu_T \sigma_R \). The confidence level is fixed at \( \alpha = 0.99 \). The Tables 6.1 and 6.2 summarize ERC and diversification results by varying the proportion \( w_i \) invested in equities but with a fixed proportion \( w_i = 0.10 \) in the riskless asset. ERC is calculated according to VaR (similar results hold for the CVaR measure). The influence of the insurance risk on the combined ERC is very small and could be neglected. The greatest influence stems from the proportion of
capital invested in equities. The diversification effect between the insurance and market risk is stable with respect to varying proportions in equities. A good approximation of the diversification effect is the stand-alone I-ERC for the second variant. These remarks remain true even if one models the insurance risk with a realistic non-normal distribution as done in Ballmann and Hürlimann(2001), Section 6.

Table 6.1: ERC and diversification for the first variant

<table>
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<tr>
<th>Parameters</th>
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<th>10</th>
<th>20</th>
<th>30</th>
<th>40</th>
<th>50</th>
</tr>
</thead>
<tbody>
<tr>
<td>$w_3$ in %</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\mu_w$ in %</td>
<td>4.8</td>
<td>5.3</td>
<td>5.8</td>
<td>6.3</td>
<td>6.8</td>
<td>7.3</td>
</tr>
<tr>
<td>$\sigma_w$ in %</td>
<td>4.5</td>
<td>4.924</td>
<td>5.635</td>
<td>6.538</td>
<td>7.566</td>
<td>8.675</td>
</tr>
</tbody>
</table>

Stand-alone ERC's

| Insurance risk | 74 | 74 | 74 | 74 | 74 | 74 |
| Market risk | 4584 | 4827 | 5404 | 6205 | 7150 | 8190 |
| Sum | 4658 | 4901 | 5478 | 6279 | 7224 | 8264 |

Allocated ERC's

| Insurance risk | -140 | -141 | -142 | -143 | -144 | -145 |
| Market risk | 4579 | 4823 | 5400 | 6201 | 7147 | 8188 |
| Aggregate risk | 4439 | 4682 | 5258 | 6058 | 7003 | 8043 |

Diversification effect

| Insurance risk | 214 | 215 | 216 | 217 | 218 | 219 |
| Market risk | 5 | 4 | 4 | 4 | 3 | 2 |
| Aggregate risk | 219 | 219 | 220 | 221 | 221 | 221 |

Table 6.2: ERC and diversification for the second variant

<table>
<thead>
<tr>
<th>Parameters</th>
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<td>7.566</td>
<td>8.675</td>
</tr>
</tbody>
</table>

Stand-alone ERC's
<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$P_1^j$</td>
<td>$X_1^j$</td>
<td>$P_2^j$</td>
<td>$X_2^j$</td>
</tr>
<tr>
<td>1</td>
<td>12.5</td>
<td>0.63</td>
<td>8.6</td>
<td>0.65</td>
</tr>
<tr>
<td>2</td>
<td>12.1</td>
<td>0.67</td>
<td>9.6</td>
<td>0.68</td>
</tr>
</tbody>
</table>

**Example 6.2**: Solvency margin and diversification for non-life insurance companies

Our model used to measure the solvency margin of non-life insurance companies is similar to the loss ratio model considered in Section 5. There are $n$ insurance companies with claims $S=(S_1,...,S_n)$, risk premiums $P=(P_1,...,P_n)$, and loss ratios $X=(X_1,...,X_n)$ with $X_i = \frac{S_i}{P_i}$, $i=1,...,n$. Given an expense ratio of 30% of the risk premium, the loss faced by company $i$ is $L_i = S_i - P_i + 0.3 \cdot P_i = P_i \cdot (X_i - 0.7)$. If the $n$ companies would merge to a single company, the loss faced by the merger is $L = P \cdot (X - 0.7)$ with $P = P_1 + ... + P_n$, $X = \sum_{i=1}^{n} w_i X_i$, $w_i = \frac{P_i}{P}$, $i=1,...,n$. In the following, the solvency margin is defined as the value-at-risk of the loss to the given confidence level $\alpha = 0.999$. Therefore, similarly to (5.3) we will consider the solvency margins $VaR_{\alpha}[L_i] = (Q_{X_i}(\alpha) - 0.7) \cdot P_i$, $i=1,...,n$, $VaR_{\alpha}[L] = (Q_{X_i}(\alpha) - 0.7) \cdot P$, as well as the diversification effect of the merger and the solvency margin allocations to the single insurance companies. The joint multivariate distribution of the loss ratios is assumed to be normally distributed with parameters $\mu, \Sigma$, which are estimated with the method of Section 5. Alternative modeling specifications of the multivariate distribution are proposed in Sections 7 to 9. For illustration, the historical data of four Kuwait insurance companies published in El-Bassiouni(1991) (see also Sundt(1992)) is used. VaR calculations are done with the risk premiums of the last period. The obtained results are summarized in the Tables 6.3 and 6.4. Risk allocation is done with the covariance principle and the Shapley principle.

**Table 6.3**: Historical data and parameters
### Parameters for companies

<table>
<thead>
<tr>
<th>Parameters</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Expected loss ratio</td>
<td>0.658</td>
<td>0.696</td>
<td>0.719</td>
<td>0.619</td>
</tr>
<tr>
<td>Standard deviation</td>
<td>0.038</td>
<td>0.069</td>
<td>0.074</td>
<td>0.102</td>
</tr>
<tr>
<td>Matrix of correlations</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>−0.327</td>
<td>0.272</td>
<td>0.311</td>
</tr>
<tr>
<td></td>
<td>−0.327</td>
<td>1</td>
<td>−0.81</td>
<td>−0.021</td>
</tr>
<tr>
<td></td>
<td>0.272</td>
<td>−0.81</td>
<td>1</td>
<td>−0.121</td>
</tr>
<tr>
<td></td>
<td>0.311</td>
<td>−0.021</td>
<td>−0.121</td>
<td>1</td>
</tr>
</tbody>
</table>

**Table 6.4**: Solvency margins and diversification for four insurance companies

<table>
<thead>
<tr>
<th>company</th>
<th>1</th>
<th>2</th>
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<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stand-alone VaR</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>In % risk premium</td>
<td>7.6</td>
<td>20.9</td>
<td>24.7</td>
<td>23.5</td>
</tr>
<tr>
<td>In Mio. Dinars</td>
<td>1244</td>
<td>2336</td>
<td>2495</td>
<td>2048</td>
</tr>
<tr>
<td>Allocated VaR</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Risk allocation with covariance principle</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>In % risk premium</td>
<td>5.5</td>
<td>−0.6</td>
<td>2.3</td>
<td>16.9</td>
</tr>
<tr>
<td>In Mio. Dinars</td>
<td>0.904</td>
<td>−0.07</td>
<td>0.229</td>
<td>1.474</td>
</tr>
<tr>
<td>Diversification effect</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>In Mio. Dinars</td>
<td>0.340</td>
<td>2.407</td>
<td>2.266</td>
<td>0.573</td>
</tr>
<tr>
<td>Allocated VaR</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Risk allocation with Shapley principle</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>In % risk premium</td>
<td>2.7</td>
<td>2.6</td>
<td>7.5</td>
<td>12.1</td>
</tr>
<tr>
<td>In Mio. Dinars</td>
<td>0.444</td>
<td>0.29</td>
<td>0.753</td>
<td>1.05</td>
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<tr>
<td>Diversification effect</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>In Mio. Dinars</td>
<td>0.800</td>
<td>2.046</td>
<td>1.742</td>
<td>0.998</td>
</tr>
</tbody>
</table>

7. **Multivariate models with arbitrary marginals.**
Our view of multivariate statistical modeling is that of Joe(1997), Section 1.7:

“Models should try to capture important characteristics, such as the appropriate density shapes for the univariate margins and the appropriate dependence structure, and otherwise be as simple as possible.”

To fulfill this, a parametric family of multivariate distributions should satisfy some desirable properties (Joe(1997), Section 4.1):

e) There should exist an interpretation like a mixture or other stochastic representation.
f) The margins, at least the univariate and bivariate ones, should belong to the same parametric family and numerical evaluation should be possible.
g) The bivariate dependence between the margins should be described by a parameter and cover a wide range of dependence.
h) The multivariate distribution and density should preferably have a closed-form representation, at least numerical evaluation should be possible.

In general, these desirable properties cannot be satisfied simultaneously. For example, multivariate normal distributions satisfy properties a), b) and c) but not d). The method of copulas, discussed in Section 7.1, satisfies property c) but implies only partial closedness under the taking of margins, and can lead to computational complexity as the dimension increases. In fact, it is an open problem to find parametric families of copulas that satisfy all of the above desirable properties (Joe(1997), Section 4.13, p.138). In the present paper such a parametric family is constructed. It is based on the method of mixtures of independent conditional distributions, discussed in Section 7.2. The important bivariate and trivariate models, together with ERC and diversification illustrations, are studied in detail in Sections 8 to 10, as well as in the Appendix.

7.1. The method of copulas.


Recall that the copula representation of a continuous multivariate distribution allows for a separate modeling of the univariate margins and the multivariate or dependence structure. Denote by \( R(F_1,\ldots,F_n) \) the class of all continuous multivariate random variables \( (X_1,\ldots,X_n) \) with given marginals \( F_i \) of \( X_i \). If \( F \) denotes the multivariate distribution of \( (X_1,\ldots,X_n) \), then the copula associated with \( F \) is a distribution function \( C: [0,1]^n \rightarrow [0,1] \) that satisfies

\[
F(x) = C(F_1(x_1),\ldots,F_n(x_n)), \quad x = (x_1,\ldots,x_n) \in R^n .
\]  

(7.1)

Reciprocally, if \( F \in R(F_1,\ldots,F_n) \) and \( F_i^{-1} \) are quantile functions of the margins, then
\[ C(u) = F(F_1^{-1}(u_1), \ldots, F_n^{-1}(u_n)), \quad u = (u_1, \ldots, u_n) \in [0,1]^n, \quad (7.2) \]

is the unique copula satisfying (7.1) (theorem of Sklar(1959)).

Copulas are especially useful for the modeling and measurement of bivariate dependence. For an axiomatic definition, one needs the important notion of concordance ordering. A copula \( C_1(u,v) \) is said to be smaller than a copula \( C_2(u,v) \) in concordance order, written \( C_1 \prec C_2 \), if one has

\[ C_1(u,v) \leq C_2(u,v), \quad (u,v) \in [0,1]^2. \quad (7.3) \]

**Definition 7.1.** (Scarsini(1984)) A numeric measure \( \kappa \), written \( \kappa_{X,Y} \) or \( \kappa_C \), of association between two continuous random variables \( X \) and \( Y \) with copula \( C(u,v) \) is a measure of concordance if it satisfies the following properties:

\begin{itemize}
  \item [(C1)] \( \kappa_{X,Y} \) is defined for every couple \( (X,Y) \) of continuous random variables
  \item [(C2)] \(-1 \leq \kappa_{X,Y} \leq 1, \quad \kappa_{X,-X} = -1, \quad \kappa_{X,X} = 1\)
  \item [(C3)] \( \kappa_{X,Y} = \kappa_{Y,X} \)
  \item [(C4)] If \( X \) and \( Y \) are independent then \( \kappa_{X,Y} = 0 \)
  \item [(C5)] \( \kappa_{-X,Y} = \kappa_{X,-Y} = -\kappa_{X,Y} \)
  \item [(C6)] If \( C_1 \prec C_2 \) then \( \kappa_{C_1} \leq \kappa_{C_2} \)
  \item [(C7)] If \( \{X_n, Y_n\} \) is a sequence of continuous random variables with copulas \( C_n \) and if \( \{C_n\} \) converges pointwise to \( C \), then \( \lim_{n \to \infty} \kappa_{C_n} = \kappa_C \)
\end{itemize}

Two famous measures of concordance, of great importance in non-parametric statistics, are Kendall’s tau

\[ \tau = 1 - 4 \cdot \int_0^1 \int_0^1 \frac{\partial}{\partial u} C(u,v) \cdot \frac{\partial}{\partial v} C(u,v) dudv, \quad (7.4) \]

and Spearman’s rho

\[ \rho_S = 12 \cdot \int_0^1 \int_0^1 \left[ C(u,v) - uv \right] dudv. \quad (7.5) \]

The latter parameter will describe completely the bivariate dependence in our construction. When extreme values are involved, tail dependence should also be measured.

**Definition 7.2.** The coefficient of (upper) tail dependence of a couple \( (X,Y) \) of continuous random variables is defined by

\[ \lambda = \lambda_{X,Y} = \lim_{\alpha \to 1^-} \Pr(Y > Q_\nu(\alpha) | X > Q_X(\alpha)), \quad (7.6) \]

provided a limit \( \lambda \) in \([0,1]\) exists. If \( \lambda \in (0,1] \) one has asymptotic dependence (in the upper tail) while if \( \lambda = 0 \) one has asymptotic independence.

Tail dependence is an asymptotic property. Its calculation follows easily from the relation
In particular, for a couple \((X,Y)\) with copula \(C(u,v)\) one has the formula

\[
\lambda = \lambda_{X,Y} = \lim_{u \to 1} \frac{1 - 2u + C(u,u)}{1 - u}.
\]  

(7.8)

**7.2. Mixtures of independent conditional distributions.**

Our goal is the construction of a parametric family of \(n\)-dimensional copulas that satisfies the desirable properties a) to d). It uses a simple variant of the method of mixtures of independent copulas to be described in Joe(1997), Section 4.5. To satisfy property b) let us focus on the \(n\) classes \(C_{ij} = C_{ij}(F_{ij}, F_{ji} \neq i)\), \(i = 1, \ldots, n\), of \(n\)-variate distributions for which the bivariate margins \(F_{ij}(x_i, x_j) = F_{ij}(x_i, x_j) = C_{ij}(F_{ij}(x_i), F_{ij}(x_j))\), \(j \neq i\), belong to a given parametric family of copulas \(C_{ij}[u_i, u_j]\). Let us assume that the conditional distributions

\[
F_{j|k}(x_j|x_i) = \frac{\partial C_{j|k}(F_{ji}(x_i), F_{ij}(x_j))}{\partial u_i}
\]  

(7.9)

are well-defined. The \(n\)-variate distribution such that the random variables \(X_j, j \neq i\), are conditionally independent given \(X_i\), is contained in \(C_{ij}\) and is defined by

\[
F^{(i)}(x) = \int \prod_{j \neq i} F_{j|x_i}^{(i)}(x_j|x_i) \, dF_{ij}(t).
\]  

(7.10)

Choosing appropriately the bivariate copulas \(C_{ij}[u_i, u_j]\), it is possible to construct \(n\)-variate copulas \(C^{(i)}(u_1, \ldots, u_n), \quad i = 1, \ldots, n\), such that \(F^{(i)}\) belongs to \(C^{(i)}\) and the bivariate margins \(F_{ij}, j \neq i\), belong to \(C_{ij}\). Moreover, any convex combination of the \(C^{(i)}\)'s, that is

\[
C(u_1, \ldots, u_n) = \sum_{i=1}^{n} \lambda_i C^{(i)}(u_1, \ldots, u_n), \quad 0 \leq \lambda_i \leq 1, \quad \sum_{i=1}^{n} \lambda_i = 1,
\]  

(7.11)

is again a \(n\)-variate copula, which by appropriate choice may satisfy the desirable properties.

**8. A bivariate model with arbitrary marginals.**

Our aim is the construction of a parametric family of \(n\)-variate copulas satisfying the four desirable properties in Section 7 and suitable for analytical ERC calculations in a variety of complex situations. Following the approach through mixtures of independent conditional distributions described in Section 7.2, it is first necessary to focus on a simple but sufficiently flexible one-parameter family of bivariate copulas, called linear Spearman copula, which is introduced and motivated in Section 8.1. The analytical evaluation of the distribution and
stop-loss transform of bivariate sums following a linear Spearman copula, required in ERC calculations, is presented in Section 8.2.

The dependence parameter of the linear Spearman copula is Spearman’s grade correlation coefficient. In practice, however, often only Pearson’s linear correlation coefficient is available. Stochastic relationships between these two parameters, which allow parameter estimation from each other, are derived in the Appendix. Finally, we illustrate in Section 8.3 the evaluation of ERC and diversification at some significant bivariate situations.

8.1. The linear Spearman copula.

We consider a one-parameter family of copulas \( C_\theta(u,v) \), which is able to model continuously a whole range of dependence between the lower Fréchet bound \( C_{-1}(u,v) = \max(u + v - 1, 0) \), the independent copula \( C_0(u,v) = uv \), and the upper Fréchet bound \( C_1(u,v) = \min(u,v) \). Such families are called inclusive or comprehensive. The extensive list by Nelsen(1999), p. 96, contains only two one-parameter inclusive families of copulas, namely those by Frank(1979) and Clayton(1978) (also called Cook-Johnson and Pareto family of copulas). Another one is the linear Spearman copula defined by

\[
C_{\theta}(u,v) = (1 - |\theta|) \cdot C_0(u,v) + |\theta| \cdot C_{\text{sgn}(\theta)}(u,v).
\] (8.1)

For \( \theta \in [0,1] \) this copula is family B11 in Joe(1997), p. 148. It represents a mixture of perfect dependence and independence. If \( X \) and \( Y \) are uniform(0,1), \( Y = X \) with probability \( \theta \) and \( Y \) is independent of \( X \) with probability \( 1 - \theta \), then \( (X,Y) \) has the linear Spearman copula. This distribution has been first considered by Konijn(1959) and motivated in Cohen(1960) along Cohen’s kappa statistic (see Hutchinson and Lai(1990), Section 10.9). For the extended copula, the chosen nomenclature linear refers to the piecewise linear sections of this copula, and Spearman refers to the fact that the grade correlation coefficient \( \rho_s \) by Spearman(1904) coincides with the parameter \( \theta \). This follows from the calculation

\[
\rho_s = 12 \cdot \int_0^1 \int_0^1 [C_\theta(u,v) - uv] \, du \, dv = \theta.
\] (8.2)

The linear Spearman copula, which leads to the so-called linear Spearman bivariate distribution, has a singular component, which according to Joe should limit its field of applicability. Despite of this it has many interesting and important properties and is suitable for computation.

For the reader’s convenience, let us describe first two extremal properties. Kendall’s \( \tau \) for this copula equals

\[
\tau = 1 - 4 \cdot \int_0^1 \int_0^1 \frac{\partial}{\partial u} C_\theta(u,v) \cdot \frac{\partial}{\partial v} C_\theta(u,v) \, du \, dv = \frac{1}{3} \rho_s \cdot [2 + \text{sgn}(\rho_s) \rho_s].
\] (8.3)

Invert this to get

\[
\rho_s = \begin{cases} 1 - \sqrt{1 + 3\tau}, & \tau \geq 0, \\ -1 + \sqrt{1 + 3\tau}, & \tau \leq 0. \end{cases}
\] (8.4)

Relate this to the convex two-parameter copula by Fréchet(1958) defined by
The linear Spearman copula satisfies the following extremal property. For $\tau \geq 0$ the upper bound for $\rho_s$ in Fréchet's copula is attained by the linear Spearman copula, and for $\tau \leq 0$ it is the lower bound, which is attained. In case $\tau \geq 0$ a second more important extremal property holds, which is related to a conjectural statement. Recall that $Y$ is stochastically increasing on $X$, written $SI(Y|X)$, if $Pr(Y > y|X = x)$ is a nondecreasing function of $x$ for all $y$. Similarly, $X$ is stochastically increasing on $Y$, written $SI(X|Y)$, if $Pr(X > x|Y = y)$ is a nondecreasing function of $y$ for all $x$. (Note that Lehmann(1966) speaks instead of positive regression dependence). If $X$ and $Y$ are continuous random variables with copula $C(u,v)$, then one has the equivalences (Nelsen(1999), Theorem 5.2.10):

\begin{align*}
SI(Y|X) & \iff \frac{\partial}{\partial u} C(u,v) \text{ is nonincreasing in } u \text{ for all } v \quad (8.7) \\
SI(X|Y) & \iff \frac{\partial}{\partial v} C(u,v) \text{ is nonincreasing in } v \text{ for all } u \quad (8.8)
\end{align*}

The Hutchinson-Lai conjecture consists of the following statement. If $(X,Y)$ satisfies the properties (8.7) and (8.8), then Spearman's $\rho_s$ satisfies the inequalities

\[ -1 + \sqrt{1 + 3\tau} \leq \rho_s \leq \min\left\{ \frac{\tau}{3}, 2\tau - \tau^2 \right\} \quad (8.9) \]

The upper bound $2\tau - \tau^2$ is attained for the one-parameter copula introduced by Kimeldorf and Sampson(1975) (see also Hutchinson and Lai(1990), Section 13.7). The lower bound is attained by the linear Spearman copula, as shown already by Konijn(1959), p. 277. Alternatively, if the conjecture holds, the maximum value of Kendall's $\tau$ by given $\rho_s$ is attained for the linear Spearman copula. Note that the upper bound $\rho_s \leq \frac{2}{\tau} \tau$ has been disproved recently by Nelsen(1999), Exercise 5.36. The remaining conjecture $-1 + \sqrt{1 + 3\tau} \leq \rho_s \leq 2\tau - \tau^2$ is still unsettled (however, see Hürlimann(2001d) for the case of bivariate extreme value copulas).

As an important modeling characteristic, let us show that the linear Spearman copula leads to a simple tail dependence structure. Using (7.8) one obtains

\[ \lambda(X,Y) = \lim_{\alpha \to \infty} \frac{1 - 2\alpha + C_0(\alpha,\alpha)}{1 - \alpha} = \lim_{\alpha \to \infty} (1 - \alpha + \theta \alpha) = \theta . \quad (8.10) \]

Therefore, unless $X$ and $Y$ are independent, a linear Spearman couple is always asymptotically dependent. This is a desirable property in insurance and financial modeling,
where data tend to be dependent in their extreme values. In contrast to this, the ubiquitous Gaussian copula yields always asymptotic independence, unless perfect correlation holds (Sibuya(1961), Resnick(1987), Chap. 5, Embrechts et al.(1998), Section 4.4).

8.2. Distribution and stop-loss transform of bivariate sums.

For the numerical computation of ERC according to the VaR and CVaR methods, it is of interest to have analytical expressions for the distribution and stop-loss transform of dependent sums \( S = X + Y \). If \((X,Y)\) follows a linear Spearman bivariate distribution, we show in Theorem 8.1 that the evaluation of these quantities depend on the knowledge of the quantiles and stop-loss transform of the independent sum of \( X \) and \( Y \), denoted \( S^\perp = X^\perp + Y^\perp \), where \((X^\perp, Y^\perp)\) represent an independent version of \((X,Y)\).

\begin{align*}
\text{Lemma 8.1.} & \quad \text{For each } L_{S_\theta}(X,Y), \quad \theta \in [-1,1], \text{ the distribution and stop-loss transform of the sum } S = X + Y \text{ satisfy the relationships} \\
& \quad \quad \quad F_S(x) = (1-|\theta|) \cdot F_{S^\perp}(x) + |\theta| \cdot F_{S_{\theta=\theta}}(x) \quad (8.11) \\
& \quad \quad \quad \pi_S(x) = (1-|\theta|) \cdot \pi_{S^\perp}(x) + |\theta| \cdot \pi_{S_{\theta=\theta}}(x) \quad (8.12) \\
\text{Proof.} & \quad \text{This follows without difficulty from the representation (8.1). } \Box
\end{align*}

\begin{align*}
\text{Lemma 8.2.} & \quad \text{Suppose } (X^+, Y^+), \text{ respectively } (X^-, Y^-), \text{ is a comonotone couple with continuous and strictly increasing marginal distributions. Then for all } u \in (0,1) \text{ one has the additive relations} \\
& \quad \quad \quad Q_{S^+}(u) = Q_X(u) + Q_Y(u), \quad Q_{S^-}(u) = Q_X(u) + Q_Y(1-u), \quad (8.13) \\
& \quad \quad \quad \pi_{S^+} \left[ Q_X(u) \right] = \pi_X \left[ Q_X(u) \right] + \pi_Y \left[ Q_Y(u) \right], \\
& \quad \quad \quad \pi_{S^-} \left[ Q_X(u) \right] = \pi_X \left[ Q_X(u) \right] + \pi_Y \left[ Q_Y(1-u) \right] - \pi_Y \left[ Q_Y(1-u) \right]. \quad (8.14) \\
\text{Proof.} & \quad \text{If } (X^+, Y^+) \text{ is a comonotone couple, it belongs to the copula } C(u,v) = \min(u,v). \text{ Inserting the expression for the conditional distribution} \\
& \quad \quad \quad F_Y|X=x(y) = \frac{\partial C}{\partial u} \left[ F_X(x), F_Y(y) \right] = 1_{\left[ Q_X(x) \leq y \right]} \text{ into the formula for the distribution of a sum} \\
& \quad \quad \quad F_{X+Y}(s) = \int_{-\infty}^{s} F_Y|X=x(s-x) dF_X(x), \quad (8.15) \\
& \quad \quad \quad \text{and making the change of variable } F_X(x) = u, \text{ one obtains}
\end{align*}
where \( u_s \) solves the equation \( Q_X(u_s) + Q_Y(u_s) = s \). Therefore (8.16) is equivalent with \( Q_{X+Y}(u_s) = Q_X(u_s) + Q_Y(u_s) \), and since \( s \) is arbitrary the first part of (8.13) is shown. The second part of (8.13) follows similarly using the copula \( C(u,v) = \max(u+v-1,0) \). To show (8.14) consider the “spread” function of a random variable \( X \) defined by

\[
T_X(u) := \pi_X[Q_X(u)] = \int_{Q_X(u)}^{\infty} (x - Q_X(u)) dF_X(x) = \int_{u}^{\infty} (Q_X(t) - Q_X(u)) dt.
\]

Using (8.13) one obtains from (8.17) immediately that

\[
T_S^+(u) = \pi_S[Q_S^+(u)] = \int_{u}^{\infty} (Q_Y(t) - Q_X(u)) dt + \int_{u}^{1} (Q_Y(1-t) - Q_Y(1-u)) dt = \pi_X[Q_X(u)] + \pi_Y[Q_Y(u)],
\]

which shows the first part of (8.14). For the second part of (8.14) one obtains similarly

\[
T_S^-(u) = \pi_S[Q_S^-(u)] = \int_{u}^{1} (Q_X(t) - Q_X(u)) dt + \int_{u}^{1} (Q_Y(1-t) - Q_Y(1-u)) dt
\]

\[
= \pi_X[Q_X(u)] + \int_{0}^{1} (Q_Y(z) - Q_Y(1-u)) dz - \int_{1-u}^{1} (Q_Y(z) - Q_Y(1-u)) dz
\]

\[
= \pi_X[Q_X(u)] + E[Y] - Q_Y(1-u) - \pi_Y[Q_Y(1-u)]
\]

The Lemma is shown. ◊

**Remark 8.1.** In case of continuous and strictly increasing margins, the first additive relations in (8.13) and (8.14) extend easily to \( n \)-variate sums \( S^+ = X^+_1 + \ldots + X^+_n \) of mutually comonotonic random variables:

\[
Q_{S^+}(u) = \sum_{i=1}^{n} Q_{X^+_i}(u), \quad \pi_{S^+}[Q_{S^+}(u)] = \sum_{i=1}^{n} \pi_{X^+_i}[Q_{X^+_i}(u)].
\]

For the quantile this is already found in Landsberger and Meilijson(1994), Denneberg(1994). Both relations are given in Dhaene et al.(2000), Kaas et al.(2000) and Hürlimann(2001a). Our approach, which has the advantage to be very elementary, yields the additional result for \( S^- \), which will be of particular use in the proof of Theorem 9.3. These relations are of great importance in ERC evaluations according to VaR and CVaR. They imply that the maximum CVaR for the aggregate loss of a portfolio \( L = (L_1, \ldots, L_n) \) with fixed marginal losses is attained at the portfolio with mutually comonotone components, and it is equal to the sum of the CVaR of its components (Hürlimann(2001a), Theorem 2.2 and 2.3) :
\[
\max \{ CVaR_\alpha[L] \} = CVaR_\alpha[L^+] = \sum_{i=1}^{n} CVaR_\alpha[L_i]. \tag{8.19}
\]

In contrast to this, the maximum VaR of a portfolio with fixed marginal losses is not attained at the portfolio with mutually components. This assertion is related to Kolmogorov’s problem treated in Makarov (1981), Rüschendorf (1982), Frank et al. (1987), Denuit et al. (1999), Durrleman et al. (2000), Luciano and Marena (2001), Cossette et al. (2001), Embrechts et al. (2001). In the comonotonic situation one has with (8.18) only the additive relation
\[
VaR_\alpha[L^+] = \sum_{i=1}^{n} VaR_\alpha[L_i] \tag{8.20}
\]

**Theorem 8.1.** For each \( LS_\theta(X,Y), \theta \in [-1,1] \), the distribution and stop-loss transform of the sum \( S = X + Y \) are determined as follows. For each \( u \in [0,1] \) one has with \( u_\theta = \frac{1}{2} \left[ 1 - \text{sgn}(\theta) \right] + \text{sgn}(\theta) \cdot u \) the formulas:

\[
F_\theta \left[ Q_X(u) + Q_Y(u_\theta) \right] = (1 - \left| \theta \right|) \cdot F_{S_\theta} \left[ Q_X(u) + Q_Y(u_\theta) \right] + \left| \theta \right| \cdot u, \tag{8.21}
\]

\[
\pi_\theta \left[ Q_X(u) + Q_Y(u_\theta) \right] = (1 - \left| \theta \right|) \cdot \pi_{S_\theta} \left[ Q_X(u) + Q_Y(u_\theta) \right] + \left| \theta \right| \cdot \left[ \pi_X \left[ Q_X(u) \right] + \text{sgn}(\theta) \cdot \pi_Y \left[ Q_Y(u_\theta) \right] + \frac{1}{2} \left[ 1 - \text{sgn}(\theta) \right] \cdot [E[Y] - Q_Y(u_\theta)] \right]. \tag{8.22}
\]

**Proof.** Apply Lemma 8.1 and 8.2. \( \Diamond \)

Though not always of simple form, analytical expressions for one of density, distribution and stop-loss transform of the independent sum \( S^\perp = X^\perp + Y^\perp \) from parametric families of margins often exist. A numerical evaluation using computer algebra systems is then easy to implement. We illustrate at the important and often encountered margins from the normal, gamma and lognormal families of distributions.

**Example 8.1:** normal marginals

Assume \( LS_\theta(X,Y) \) has normal margins \( X \sim N(\mu_X, \sigma_X^2), Y \sim N(\mu_Y, \sigma_Y^2) \). Then \( S = X + Y \) has mean \( \mu = \mu_X + \mu_Y \) and variance \( \sigma^2 = \sigma_X^2 + 2\theta \sigma_X \sigma_Y + \sigma_Y^2 \). The independent and comonotonic sums \( S^\perp \sim N(\mu, \sigma^2), S^\perp \sim N(\mu, \sigma^2) \) have the same mean but different variances \( \sigma^2 = \sigma_X^2 + \sigma_Y^2 \), \( \sigma^2 = (\sigma_X + \sigma_Y)^2 \). To implement the formulas (8.21) and (8.22) note that if \( X \sim N(\mu, \sigma^2) \) then \( Q_X(u) = \mu + \sigma \Phi^{-1}(u) \) and \( \pi_X(x) = \sigma \phi \left( \frac{x - \mu}{\sigma} \right) - \left( x - \mu \right) \left( 1 - \Phi \left( \frac{x - \mu}{\sigma} \right) \right) \), with \( \Phi(x) \) the standard normal distribution and \( \phi(x) = \Phi'(x) \). In case \( \theta \neq -1, 0, 1 \), \( F_\theta(x) \) differs from \( \Phi \left( \frac{x - \mu}{\sigma} \right) \) the distribution of the sum from a bivariate normal couple with correlation \( \theta \) .
Example 8.2: gamma marginals

Assume \( \theta_{LS}(X,Y) \) has gamma margins \( X \sim Ga(\alpha_X, \beta_X), \ Y \sim Ga(\alpha_Y, \beta_Y) \). Then one has \( Q_X(u) = \beta_X^{-1} Q_{Ga(\alpha_X, \beta_X)}(u) \) and \( Q_Y(u) = \beta_Y^{-1} Q_{Ga(\alpha_Y, \beta_Y)}(u) \), where \( Q_{Ga(\alpha, \beta)}(u) \) is an implemented function. There are several available analytical expressions for the distribution of the independent sum. Johnson et al. (1994), pp. 384-85, mention Mathai (1982), Moschopoulos (1985) and Sim (1992). One can add Provost (1989) and Hürlimann (2001a). The latter author shows the formula:

\[
F_{S^+}(x) = \left( \frac{\beta_Y}{\beta_X} \right)^{\alpha_X} \sum_{j=0}^{\infty} \frac{(1 - \frac{\beta_Y}{\beta_X})^j \Gamma(\alpha_Y + j)}{j! \Gamma(\alpha_Y)} \Gamma(\beta_X x, \alpha_X + \alpha_Y + j),
\]

(8.23)

where \( \beta_Y \geq \beta_X \), \( \Gamma(\alpha) \) is the gamma function, and \( \Gamma(x, \alpha) \) is the distribution function of \( Ga(\alpha, 1) \). Similarly, an analytical expression for the stop-loss transform is

\[
\pi_{S^+}(x) = \mu - x(1 - F_{S^+}(x)) - \frac{1}{\beta_X \beta_Y} \sum_{j=0}^{\infty} \frac{(1 - \frac{\beta_Y}{\beta_X})^j \Gamma(\alpha_Y + j)(\alpha_X + \alpha_Y + j)}{j! \Gamma(\alpha_Y)} \Gamma(\beta_X x, \alpha_X + \alpha_Y + j + 1),
\]

(8.24)

Moreover, if \( X \sim Ga(\alpha_X, \beta_X) \) one has

\[
\pi_X(x) = \mu_X (1 - \Gamma(\beta_X x, \alpha_X + 1)) - x(1 - \Gamma(\beta_X x, \alpha_X)), \quad \mu_X = \frac{\alpha_X}{\beta_X}.
\]

(8.25)

Example 8.3: lognormal marginals

Since \( \theta_{LS}(X,Y) \) has lognormal margins one has \( X \sim \ln N(\alpha_X, \beta_X), \ Y \sim \ln N(\alpha_Y, \beta_Y) \). Since the distribution and quantile functions are usually implemented, it remains to implement \( F_{S^+}(x), \pi_{S^+}(x) \) and \( \pi_X(x) \). From Johnson et al. (1994), p. 218, one borrows the analytical expression for the density of \( S^+ \):

\[
f_{S^+}(x) = \frac{1}{2\pi \beta_X \beta_Y} \int_0^1 \frac{1}{t(1-t)} \exp \left\{ -\frac{1}{2} \left( \frac{\ln(1-t) + \ln(x) - \alpha_X}{\beta_X} \right)^2 - \frac{1}{2} \left( \frac{\ln(t) + \ln(x) - \alpha_Y}{\beta_Y} \right)^2 \right\} dt
\]

(8.26)

Assuming finite integrals are implemented, one obtains further

\[
F_{S^+}(x) = \int_0^x f_{S^+}(y) dy, \quad \pi_{S^+}(x) = \mu - x + \int_0^x (x-y) f_{S^+}(y) dy
\]

(8.27)

Moreover, if \( X \sim \ln N(\alpha_X, \beta_X) \) one has

\[
\pi_X(x) = \mu_X (1 - \Phi(\frac{\ln(x) - \alpha_X}{\beta_X})) - x(1 - \Phi(\frac{\ln(x) - \alpha_X}{\beta_X})), \quad \mu_X = \exp(\alpha_X + \frac{1}{2} \beta_X^2).
\]

(8.28)
8.3. ERC and diversification for bivariate portfolio models.

We illustrate the exact numerical evaluation of ERC and diversification effects at two simple insurance and market bivariate portfolio models.

Example 8.4: Insurance bivariate portfolio model

In the context of Section 5 consider a bivariate portfolio consisting of a low and high insurance risk whose loss ratios $X_1$ and $X_2$ have means $\mu_1=0.70, \mu_2=0.60$ and standard deviations $\sigma_1=0.10, \sigma_2=0.20$. The Pearson linear correlation coefficient between $X_1$ and $X_2$ is $\rho=0.6$. For a risk premium $P=100$ let us invest the proportion $0 \leq w_1 \leq 1$ in the low risk and the proportion $w_2=1-w_1$ in the high risk. The loss ratio of the portfolio is described by the bivariate sum $X = w_1X_1 + w_2X_2$. We assume that $(X_1, X_2)$ is a linear Spearman couple $LS_\theta(X_1, X_2)$ with gamma margins, where $\theta=0.60239$ has been calculated according to the formula (A.21) of Example A.2 in the Appendix. The confidence level is fixed at $\alpha=0.099$. The Table 8.1 summarizes ERC and diversification results by fixed proportion $w_1=0.75$ invested in the low risk for the VaR and CVaR measures. The results obtained for $\theta=0$ (independent risks) and $\theta=1$ (comonotonic risks) are listed for comparison. By varying $\theta \in [0,1]$ they correspond to the minimal and maximal values of ERC and are obtained with the method described in Hürlimann(2001a).

Table 8.1: ERC and diversification for insurance bivariate portfolio model

<table>
<thead>
<tr>
<th>Stand-alone ERC</th>
<th>VaR method</th>
<th>CVaR method</th>
</tr>
</thead>
<tbody>
<tr>
<td>Low risk</td>
<td>19.0</td>
<td>22.2</td>
</tr>
<tr>
<td>High risk</td>
<td>14.0</td>
<td>16.7</td>
</tr>
<tr>
<td>Sum</td>
<td>33.0</td>
<td>38.9</td>
</tr>
<tr>
<td>Spearman $\theta$</td>
<td>0</td>
<td>0.60239</td>
</tr>
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<td></td>
<td>0</td>
<td>0.60239</td>
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<table>
<thead>
<tr>
<th>Allocated ERC</th>
<th>covariance principle</th>
</tr>
</thead>
<tbody>
<tr>
<td>Low risk</td>
<td>18.8 19.0 18.4 18.9 22.2</td>
</tr>
<tr>
<td>High risk</td>
<td>11.3 14.0 8.2 11.4 16.7</td>
</tr>
<tr>
<td>Portfolio risk</td>
<td>30.1 33.0 26.6 30.3 38.9</td>
</tr>
</tbody>
</table>

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<thead>
<tr>
<th>Diversification effect</th>
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<tr>
<td>Low risk</td>
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<tr>
<td>High risk</td>
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<tr>
<td>Portfolio risk</td>
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<tr>
<td>Allocated ERC</td>
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<td>Low risk</td>
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<td>Portfolio risk</td>
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<th>Diversification effect</th>
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<td>Low risk</td>
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**Example 8.5:** Market bivariate portfolio model

Consider the market ERC model of Section 2.2 with only two risky investments. There is an investment opportunity in a stock market with expected accumulated return \( \mu_1 = 1.10 \), standard deviation \( \sigma_1 = 0.15 \), and an investment opportunity in a bond market with expected accumulated return \( \mu_2 = 1.05 \), standard deviation \( \sigma_2 = 0.05 \). If \( 0 \leq w_1 \leq 1, \ w_2 = 1 - w_1 \), are the weights in each asset category, the portfolio accumulated return \( R_p = w_1 R_1 + w_2 R_2 \) is modelled using a linear Spearman couple \( LS_\rho(R_1, R_2) \) with arbitrary marginals but positive Spearman coefficient \( \theta \in [0,1] \) (typical for market risks). The ERC formulas (2.4) and (2.5) are evaluated and compared for normal, gamma and lognormal margins by fixed means and variances. These models are denoted LSN, LST and LSlnN. The evaluation for the usual bivariate normal, denoted BN, is also performed. For the fixed confidence level \( \alpha = 0.99 \) and the weight \( w_1 = 0.25 \), the obtained figures by varying Spearman coefficient are reported in Table 8.2. For these models the maximum ERC is attained at the bivariate Spearman normal, and the minimum ERC is attained at the bivariate Spearman lognormal.

**Table 8.2:** ERC for market bivariate portfolio model

<table>
<thead>
<tr>
<th>\theta</th>
<th>LSN/LN</th>
<th>LST/\Gamma</th>
<th>BN</th>
<th>LSN/LN</th>
<th>LSN/LN</th>
<th>LST/\Gamma</th>
<th>BN</th>
<th>LSN/LN</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>5.4</td>
<td>5.5</td>
<td>6.1</td>
<td>6.1</td>
<td>6.9</td>
<td>7.1</td>
<td>7.9</td>
<td>7.9</td>
</tr>
<tr>
<td>0.25</td>
<td>7.0</td>
<td>7.3</td>
<td>7.5</td>
<td>8.0</td>
<td>9.1</td>
<td>9.5</td>
<td>9.6</td>
<td>10.5</td>
</tr>
<tr>
<td>0.50</td>
<td>8.2</td>
<td>8.6</td>
<td>8.9</td>
<td>9.4</td>
<td>10.4</td>
<td>10.9</td>
<td>11.1</td>
<td>12.0</td>
</tr>
<tr>
<td>0.75</td>
<td>9.1</td>
<td>9.5</td>
<td>10.1</td>
<td>10.4</td>
<td>11.1</td>
<td>11.7</td>
<td>12.4</td>
<td>13.0</td>
</tr>
<tr>
<td>1</td>
<td>9.7</td>
<td>10.2</td>
<td>11.2</td>
<td>11.2</td>
<td>11.7</td>
<td>12.3</td>
<td>13.7</td>
<td>13.7</td>
</tr>
</tbody>
</table>

9. **A trivariate model with arbitrary marginals.**

Applying the method of mixtures of independent conditional distributions described in Section 7.2, we construct a family of 3-dimensional copulas \( C(u_1, u_2, u_3) \) that satisfies the four desirable properties mentioned in Section 7, and therefore provides a first answer to the
open problem 4.1 in Joe(1997), p.138. The bivariate copulas \( C_y[u_i,u_j] \), which describe completely the bivariate margins, are the linear Spearman copulas (8.1) with parameters \( \theta_y \in [-1,1] \). In the Sections 9.1 to 9.3 we distinguish between positive, negative and mixed positive-negative dependence models. Section 9.4 describes the evaluation of the distribution and stop-loss transform for trivariate sums from these models. Applications to ERC and diversification follow in Section 10.

9.1. A trivariate model with positive dependence.

For \( i, j \in \{1,2,3\}, j \neq i \), let \( C_y[u_i,u_j] = (1-\theta_y)u_iu_j + \theta_y \min(u_i,u_j) \) be the linear Spearman copula with parameter \( \theta_y \in [0,1] \), where by symmetry \( \theta_{ji} = \theta_y \). To evaluate \( F^{(1)}(x) \) as defined in (7.10) one needs the conditional distributions (7.9) given by

\[
F_{2k}(x_2|x_1) = (1-\theta_{12}) \cdot F_2(x_2) + \theta_{12} \cdot 1_{[x_2 \leq \theta_{12} F_2(x_1)]},
\]

\[
F_{3k}(x_3|x_1) = (1-\theta_{13}) \cdot F_3(x_3) + \theta_{13} \cdot 1_{[x_3 \leq \theta_{13} F_3(x_1)]}.
\]

Inserted in (7.10) one obtains in a straightforward way

\[
F^{(1)}(x) = \int_{-\infty}^{x_1} F_{2k}(x_2|x_1) F_{3k}(x_3|x_1) dF_1(x_1)
\]

\[
= (1-\theta_{12}) \left[ (1-\theta_{13}) F_1(x_1) F_2(x_2) F_3(x_3) + \theta_{13} F_2(x_2) \min\{F_1(x_1), F_3(x_3)\} \right] + \theta_{12} \left[ (1-\theta_{13}) F_1(x_1) \min\{F_2(x_2), F_3(x_3)\} \right] + \theta_{13} \left[ \theta_{12} \min\{F_1(x_1), F_2(x_2), F_3(x_3)\} \right].
\]

This trivariate distribution belongs to the 3-dimensional copula

\[
C^{(1)}(u_1,u_2,u_3) = (1-\theta_{12}) \left[ (1-\theta_{13}) u_1 u_2 u_3 + \theta_{13} u_2 \min\{u_1,u_3\} \right] + \theta_{12} \left[ (1-\theta_{13}) u_1 \min\{u_1,u_2\} + \theta_{13} \min\{u_1,u_2,u_3\} \right]
\]

Through permutation of the indices one obtains similarly trivariate distributions \( F^{(2)} \) and \( F^{(3)} \) with copulas

\[
C^{(2)}(u_1,u_2,u_3) = (1-\theta_{12}) \left[ (1-\theta_{23}) u_1 u_2 u_3 + \theta_{23} u_1 \min\{u_2,u_3\} \right] + \theta_{12} \left[ (1-\theta_{23}) u_1 \min\{u_1,u_2\} + \theta_{23} \min\{u_1,u_2,u_3\} \right]
\]

\[
C^{(3)}(u_1,u_2,u_3) = (1-\theta_{23}) \left[ (1-\theta_{13}) u_1 u_2 u_3 + \theta_{13} u_2 \min\{u_1,u_3\} \right] + \theta_{23} \left[ (1-\theta_{13}) u_1 \min\{u_2,u_3\} + \theta_{13} \min\{u_1,u_2,u_3\} \right].
\]

It follows immediately that the bivariate margins \( F_y(x_i,x_j) \) of \( F^{(k)} \), \( i,j,k \in \{1,2,3\} \), belong to linear Spearman copulas, whose Spearman correlation coefficients \( \rho_y^5 \) are described in tabular form as follows:
For each $i = 1, 2, 3$ it follows that $F^{(i)}$ belongs to the class $\mathcal{C}_l(F_j, j \neq i)$, with the desired linear Spearman bivariate margins $F_j$, $j \neq i$. Unfortunately, the third linear Spearman bivariate margin $F_{jk}$, $j, k \neq i$, has the Spearman correlation coefficient $\rho_{jk}^S = \theta_{jk}$, which in general differs from the parameter $\theta_{jk}$. To construct a trivariate distribution $F$, whose linear Spearman bivariate margins may have general Spearman correlation coefficients $\rho_{jk}^S \in [0, 1]$, we consider the convex combination of the $C^{(i)}$s, $i = 1, 2, 3$, defined for all $\theta = (\theta_{12}, \theta_{13}, \theta_{23}) \neq (1, 1, 1)$ by

$$
C(u_1, u_2, u_3) = \frac{1}{3 - (\theta_{12} + \theta_{13} + \theta_{23})} \left[ \frac{(1 - \theta_{23})C^{(1)}(u_1, u_2, u_3) + (1 - \theta_{13})C^{(2)}(u_1, u_2, u_3) + (1 - \theta_{12})C^{(3)}(u_1, u_2, u_3)}{	heta_{12} + \theta_{13} + \theta_{23}} \right]
$$

$$
= \frac{1}{3 - (\theta_{12} + \theta_{13} + \theta_{23})} \left[ \begin{array}{l}
3(1 - \theta_{12})(1 - \theta_{13})(1 - \theta_{23})u_1u_2u_3 \\
+ 2(1 - \theta_{12})(1 - \theta_{13})\theta_{23}u_1 \min(u_2, u_3) \\
+ 2(1 - \theta_{12})(1 - \theta_{23})\theta_{13}u_2 \min(u_1, u_3) \\
+ 2(1 - \theta_{13})(1 - \theta_{23})\theta_{12}u_3 \min(u_1, u_2) \\
+ (\theta_{12}\theta_{23} + \theta_{13}\theta_{23} + \theta_{12}\theta_{13} - 3\theta_{12}\theta_{13}\theta_{23}) \min(u_1, u_2, u_3)
\end{array} \right] \tag{9.7}
$$

If $\theta = (\theta_{12}, \theta_{13}, \theta_{23}) = (1, 1, 1)$ one sets $C(u_1, u_2, u_3) = \min(u_1, u_2, u_3)$ the copula belonging to a comonotone triple. By construction, the trivariate distribution $F(x) = C[F_1(x_1), F_2(x_2), F_3(x_3)]$ has linear Spearman bivariate margins $F_j(x_j) = C^{(j)}[F_j(x_1), F_j(x_2), F_j(x_3)]$ where the coefficients $\rho_{ij}^S$ in $C^{(j)}[u_1, u_j] = (1 - \rho_{ij}^S)u_iu_j + \rho_{ij}^S \min(u_i, u_j)$ satisfy the relationships

$$
\rho_{12}^S = \frac{(1 - \theta_{12})\theta_{13}\theta_{23} + \theta_{13}(2 - \theta_{13} - \theta_{23})}{3 - (\theta_{12} + \theta_{13} + \theta_{23})},
\rho_{13}^S = \frac{(1 - \theta_{13})\theta_{12}\theta_{23} + \theta_{12}(2 - \theta_{12} - \theta_{23})}{3 - (\theta_{12} + \theta_{13} + \theta_{23})},
\rho_{23}^S = \frac{(1 - \theta_{23})\theta_{12}\theta_{13} + \theta_{23}(2 - \theta_{23} - \theta_{13})}{3 - (\theta_{12} + \theta_{13} + \theta_{23})}. \tag{9.8}
$$

Though it is not a priori guaranteed that the non-linear trivariate function (9.8), which maps $\theta = (\theta_{12}, \theta_{13}, \theta_{23}) \in [0, 1]^3 - \{1, 1, 1\}$ to $\rho^S = (\rho_{12}^S, \rho_{13}^S, \rho_{23}^S) \in [0, 1]^3 - \{1, 1, 1\}$, is one-to-one, the obtained copula (9.7) is sufficiently general and simple to yield tractable positive dependent trivariate distributions with bivariate margins equal or at least close to given linear Spearman bivariate margins. By appropriate choice of the univariate margins, say gamma or lognormal margins, the obtained parametric family of trivariate copulas satisfies the four desirable properties in Section 7.
9.2. A trivariate model with negative dependence.

For \( i, j \in \{1, 2, 3\}, j \neq i \), let \( C_{ij}(u_i, u_j) = (1 + \theta_{ij})u_iu_j - \theta_{ij} \min(u_i + u_j - 1, 0) \) be the linear Spearman copula with parameter \( \theta_{ij} \in [-1, 0] \), where by symmetry \( \theta_{ji} = \theta_{ij} \). Proceeding as in Section 9.1, one obtains trivariate distributions \( F^{(i)}(x), i = 1, 2, 3 \), which belong to the 3-dimensional copulas \( \{0,1\}, \{\max(0,1)\}, \min(0,1\} \).

\[
C^{(1)}(u_1, u_2, u_3) = (1 + \theta_{12})[(1 + \theta_{13})u_1u_2u_3 - \theta_{13}u_1 \max[u_1 + u_3 - 1, 0]] \\
- \theta_{12}[(1 + \theta_{13})u_2 \max[u_2 + u_3 - 1, 0] - \theta_{13} \max[u_1 + \min(u_2, u_3) - 1, 0]],
\]

\[
C^{(2)}(u_1, u_2, u_3) = (1 + \theta_{12})[(1 + \theta_{23})u_1u_2u_3 - \theta_{23}u_1 \max[u_2 + u_3 - 1, 0]] \\
- \theta_{12}[(1 + \theta_{23})u_2 \max[u_2 + u_3 - 1, 0] - \theta_{23} \max[u_1 + \min(u_2, u_3) - 1, 0]],
\]

\[
C^{(3)}(u_1, u_2, u_3) = (1 + \theta_{12})[(1 + \theta_{13})u_1u_2u_3 - \theta_{13}u_2 \max[u_1 + u_3 - 1, 0]] \\
- \theta_{12}[(1 + \theta_{13})u_1 \max[u_1 + u_3 - 1, 0] - \theta_{13} \max[u_2 + \min(u_1, u_3) - 1, 0]].
\]

(9.9)

Again, the bivariate margins \( F_{ij} \) of \( F^{(k)} \) belong to linear Spearman copulas with Spearman correlation coefficients \( \rho^S_{ij} \) as described in Section 9.1. For the reasons explained there we consider the convex combination of these copulas defined for all \( \theta = (\theta_{12}, \theta_{13}, \theta_{23}) \neq (-1,-1,-1) \) by

\[
C(u_1, u_2, u_3) = \frac{1}{3 + \theta_{12} + \theta_{13} + \theta_{23}} \left[ (1 + \theta_{23})C^{(1)}(u_1, u_2, u_3) + (1 + \theta_{13})C^{(2)}(u_1, u_2, u_3) + (1 + \theta_{12})C^{(3)}(u_1, u_2, u_3) \right]
\]

\[
= \frac{1}{3 + \theta_{12} + \theta_{13} + \theta_{23}} \left[ 3(1 + \theta_{12})(1 + \theta_{13})(1 + \theta_{23})u_1u_2u_3 \\
- 2(1 + \theta_{12})(1 + \theta_{13})\theta_{23}u_1 \max(u_2 + u_3 - 1, 0) \\
- 2(1 + \theta_{12})(1 + \theta_{23})\theta_{13}u_2 \max(u_1 + u_3 - 1, 0) \\
+ (1 + \theta_{23})\theta_{12}u_3 \max(u_1 + \min(u_2, u_3) - 1, 0) \\
+ (1 + \theta_{13})\theta_{12}u_3 \max(u_2 + \min(u_1, u_3) - 1, 0) \right].
\]

(9.10)

If \( \theta = (\theta_{12}, \theta_{13}, \theta_{23}) = (-1,-1,-1) \) one sets

\[
C(u_1, u_2, u_3) = \frac{1}{3} \left[ \max(u_1 + \min(u_2, u_3) - 1, 0) + \max(u_2 + \min(u_1, u_3) - 1, 0) \\
+ \max(u_3 + \min(u_1, u_2) - 1, 0) \right].
\]

(9.11)

The latter copula puts the same weight on each of three “minimal” trivariate distributions of the type studied by Ruiz-Rivas(1979). In probabilistic terms, the trivariate distribution \( \max(F_1(x_1) + \min(F_2(x_2), F_3(x_3)) - 1, 0) \) assigns probability one to the stochastic curve \( F_1(X_1) = F_2(X_2) = F_3(X_3) \) meaning that the random triple \( (-X_1, X_2, X_3) \) is a comonotone triple. Similarly to Section 9.1, any trivariate distribution belonging to the copula (9.10) has linear Spearman bivariate margins, whose Spearman correlation coefficients \( \rho^S_{ij} \) satisfy relationships similar to (9.8), and the same concluding remarks apply.
9.3. A trivariate model with positive-negative dependence.

In practice, random triples may have bivariate components displaying positive and negative dependence. It suffices to distinguish between the two cases \( \theta_{12}, \theta_{13} > 0, \theta_{23} < 0 \) and \( \theta_{12} > 0, \theta_{13}, \theta_{23} < 0 \). Proceeding as in Sections 9.1 and 9.2 but leaving the straightforward details to the interested reader, we list only the obtained final convex combinations of copulas.

Case 1: \( \theta_{12}, \theta_{13} > 0, \theta_{23} < 0 \)

For all \( \theta = (\theta_{12}, \theta_{13}, \theta_{23}) \neq (1,1,-1) \) one obtains

\[
C(u_1, u_2, u_3) = \frac{1}{3 - \theta_{12} - \theta_{13} + \theta_{23}} \left[ 3(1 - \theta_{12})(1 - \theta_{13})(1 + \theta_{23})u_1u_2u_3 - 2(1 - \theta_{12})(1 - \theta_{13})\theta_{23}u_1 \max(u_2 + u_3 - 1, 0) + 2(1 - \theta_{12})(1 + \theta_{23})\theta_{13}u_2 \min(u_1, u_3) + 2(1 - \theta_{13})(1 + \theta_{23})\theta_{12}u_3 \min(u_1, u_2) - (1 - \theta_{12})\theta_{13}\theta_{23} \max(u_2 + \min(u_1, u_3) - 1, 0) - (1 - \theta_{13})\theta_{12}\theta_{23} \max(u_3 + \min(u_1, u_2) - 1, 0) + (1 + \theta_{23})\theta_{12}\theta_{13} \min(u_1, u_2, u_3) \right] \tag{9.12} \]

In the extreme case \( \theta = (\theta_{12}, \theta_{13}, \theta_{23}) = (1,1,-1) \) one sets

\[
C(u_1, u_2, u_3) = \frac{1}{3} \left[ \max(u_2 + \min(u_1, u_3) - 1, 0) + \max(u_3 + \min(u_1, u_2) - 1, 0) \right] + \min(u_1, u_2, u_3) \tag{9.13} \]

Case 2: \( \theta_{12} > 0, \theta_{13}, \theta_{23} < 0 \)

For all \( \theta = (\theta_{12}, \theta_{13}, \theta_{23}) \neq (1,-1,-1) \) one obtains

\[
C(u_1, u_2, u_3) = \frac{1}{3 - \theta_{12} + \theta_{13} + \theta_{23}} \left[ 3(1 - \theta_{12})(1 + \theta_{13})(1 + \theta_{23})u_1u_2u_3 - 2(1 - \theta_{12})(1 + \theta_{13})\theta_{23}u_1 \max(u_2 + u_3 - 1, 0) - 2(1 - \theta_{13})(1 + \theta_{23})\theta_{13}u_2 \min(u_1, u_3) + 2(1 + \theta_{12})\theta_{13}\theta_{23} \max(u_1, u_2) + [1 - \theta_{12} - \theta_{13} - \theta_{23}] \max(u_1 + \min(u_1, u_2) - 1, 0) \right] \tag{9.14} \]

In the extreme case \( \theta = (\theta_{12}, \theta_{13}, \theta_{23}) = (1,-1,-1) \) one sets

\[
C(u_1, u_2, u_3) = \max(u_3 + \min(u_1, u_2) - 1, 0). \tag{9.15} \]
9.4. Distribution and stop-loss transform of trivariate sums.

The analytical exact evaluation of the distribution and stop-loss transform of dependent sums \( S = X + Y + Z \) from a trivariate model in Section 9 is more complex. In general, suppose a \( n \)-dimensional copula is a convex combination of other copulas, say \( C = \sum \lambda_j C^j \).

Then the distribution \( F_S(s) \) and stop-loss transform \( \pi_S(s) \) of dependent sums \( S = \sum_{i=1}^{n} X_i \) from the multivariate model with copula \( C \) are the convex combinations of the distributions \( F_{S^j}(s) \) and stop-loss transform \( \pi_{S^j}(s) \) of the dependent sums \( S^j = \sum_{i=1}^{n} X^j_i \) from the multivariate models with copulas \( C^j \), that is \( F_S(s) = \sum \lambda_j F_{S^j}(s) \) and \( \pi_S(s) = \sum \lambda_j \pi_{S^j}(s) \). Since this result applies to the trivariate copulas of Section 9, it suffices, up to permutations of variables, to discuss the evaluation of the distribution and stop-loss transform of sums from a trivariate model with one of the following five elementary copulas \( C^1(u,v,w) = uvw \), \( C^2(u,v,w) = \min(u,v,w) \), \( C^3(u,v,w) = u \cdot \min(v,w) \), \( C^4(u,v,w) = u \cdot \max(v + w - 1,0) \), \( C^5(u,v,w) = \max(u + \min(v,w) - 1,0) \).

A trivariate distribution with copula \( C^1 \) belongs to a random triple \((X,Y,Z)\) with independent components, while a distribution with copula \( C^2 \) belongs to a random triple with comonotone components. For \( C^1 \) the distribution and stop-loss transform of sums are obtained using convolution formulas, while for \( C^2 \) they are obtained through addition of the same quantities from the individual components as stated in Remark 8.1. For example, the case of gamma marginals has been thoroughly discussed in Hürlimann(2001a). There remains the derivation of summation formulas for the other three copulas. We restrict the attention to non-negative random variables with continuous and strictly increasing distributions whose densities exist.

**Theorem 9.1.** Suppose \((X,Y,Z)\) is a random triple whose trivariate distribution belongs to the copula \( C^3(u,v,w) = u \cdot \min(v,w) \). Assume the continuous and strictly increasing marginal distributions \( F_X(x), F_Y(y), F_Z(z) \) with support \([0,\infty) \) have densities \( f_X(x), f_Y(y), f_Z(z) \). Then the distribution and stop-loss transform of the sum \( S = X + Y + Z \) are determined by

\[
F_S(s) = \int_0^s u f_X[s - Q_X(u) - Q_Z(u)] \left[ f_Y[Q_Y(u)]^{-1} + f_Z[Q_Z(u)]^{-1} \right] du,
\]

\[
\pi_S(s) = E[S] - s + \int_0^s u f_X[s - Q_X(u) - Q_Z(u)] \left[ f_Y[Q_Y(u)]^{-1} + f_Z[Q_Z(u)]^{-1} \right] du,
\]

where \( u_s \) solves the equation

\[
Q_Y(u_s) + Q_Z(u_s) = s.
\]

**Proof.** Using Dhaene and Goovaerts(1996), Lemma 2, one obtains the formulas
\[ \pi_s(s) = E[S] - s + I(s), \quad F_s(s) = 1 + \frac{d}{ds} \pi_s(s) = I'(s), \] with

\[ I(s) = \int_0^s F_{(X,Y+Z)}(x,s-x)dx. \quad (9.19) \]

Furthermore the distribution in (9.19) can be calculated from

\[ F_{(X,Y+Z)}(x,w) = \int_0^x F_{Y+Z}(w) F_X(t) \, dt. \quad (9.20) \]

The form of the copula \( C_3 \) implies that \( X \) is independent from \((Y,Z)\) and \( F_{(Y,Z)}(y,z) = F_{(Y,Z)}(y) F_{(Y,Z)}(z) \), hence

\[ F_{Y+Z}(w) = F_Y(w), \quad (Y,Z) \] is a comonotone couple. It follows from (9.20) that \( F_{(X,Y+Z)}(x,w) = F_X(x) \cdot F_{Y+Z}(w) \). Inserting in (9.19) and making the change of variable \( F_{Y+Z}(t) = u \), one obtains successively

\[
I(s) = \int_0^s F_X(s-t) F_{Y+Z}(t) dt = \int_0^{F_{Y+Z}(s)} u F_X[s - Q_{Y+Z}(u)] \cdot Q_{Y+Z}'(u) du \\
= \int_0^{F_{Y+Z}(s)} u F_X[s - Q_Y(u) - Q_Z(u)] \cdot \left[ f_Y[Q_Y(u)]^{-1} + f_Z[Q_Z(u)]^{-1} \right] du,
\]

where the last equality follows because \((Y,Z)\) is a comonotone couple and by definition of \( u_s \) in (9.18). The formula (9.17) is shown. The formula (9.16) follows from

\[
F_X(s) = I'(s) = F_X(0) \cdot F_{Y+Z}(s) + \int_0^s f_X(s-t) F_{Y+Z}(t) dt = \int_0^s f_X(s-t) F_{Y+Z}(t) dt
\]

making the same change of variable \( F_{Y+Z}(t) = u \).

**Theorem 9.2.** Suppose \((X,Y,Z)\) is a random triple whose trivariate distribution belongs to the copula \( C_4(u,v,w) = u \cdot \max(v+w-1,0) \). Under the assumptions of Theorem 9.1 the distribution and stop-loss transform of the sum \( S = X + Y + Z \) are determined by

\[
F_S(s) = \int_0^{u_s} u f_X[s - Q_Y(u) - Q_Z(1-u)] \cdot \left[ f_Y[Q_Y(u)]^{-1} - f_Z[Q_Z(1-u)]^{-1} \right] du, \quad (9.21)
\]

\[
\pi_S(s) = E[S] - s + \int_0^{u_s} u f_X[s - Q_Y(u) - Q_Z(1-u)] \cdot \left[ f_Y[Q_Y(u)]^{-1} - f_Z[Q_Z(1-u)]^{-1} \right] du, \quad (9.22)
\]

where \( u_s \) solves the equation
\[ Q_y(u_y) + Q_z(1-u_z) = s. \]  \hfill (9.23)

**Proof.** The derivation is similar to the proof of Theorem 9.1. The only difference is that the distribution of the couple \((Y, Z)\) is \(F_{Y,Z}(y,z) = \max\{F_y(y) + F_z(z) - 1, 0\}\), which means that \((Y, -Z)\) is a comonotone couple. The remaining details follow using the second part of (8.13) in Lemma 8.2. ◊

**Theorem 9.3.** Suppose \((X, Y, Z)\) is a random triple whose trivariate distribution belongs to the copula \(C^5(u,v,w) = \max(u + \min(v, w) - 1, 0)\). Under the assumption of continuous and strictly increasing marginal distributions the quantile and stop-loss transform of the sum \(S = X + Y + Z\) satisfy the additive relationships

\[ Q_y(u) = Q_x(1-u) + Q_y(u) + Q_z(u), \]
\[ \pi_y[Q_y(u)] = \pi_x[Q_x(u)] + \pi_y[Q_y(u)] + \mathbb{E}[X] - Q_X(1-u) - \pi_x[Q_x(1-u)] \]

**Proof.** The form of the copula \(C^5\) implies that \((Y, Z, -X)\) is a comonotone triple, hence \((Y + Z, -X)\) and \((Y, Z)\) are comonotone couples. The result follows by an application of the four formulas in (8.13) and (8.14). ◊

10. ERC and diversification for trivariate portfolio models.

We illustrate the exact and approximate numerical evaluation of ERC and diversification effects for the trivariate model of Section 9 at a simple insurance portfolio model.

**Example 10.1:** Insurance trivariate portfolio model

Consider the trivariate extension of the insurance bivariate portfolio model studied in Example 8.4. Suppose the portfolio consists of a low, moderate and high insurance risk whose loss ratios \(X_1, X_2\) and \(X_3\) have means \(\mu_1 = 0.70, \mu_2 = 0.65, \mu_3 = 0.60\) and standard deviations \(\sigma_1 = 0.10, \sigma_2 = 0.15, \sigma_3 = 0.20\). The Pearson linear correlation coefficients \(\rho_{ij}\) between \(X_i\) and \(X_j\) are \(\rho_{12} = 0.75, \rho_{13} = 0.60, \rho_{23} = 0.50\). For a risk premium \(P = 100\) the proportions invested in the low, moderate and high risk are \(w_1 = 0.40, w_2 = 0.20, w_3 = 0.40\). The loss ratio of the portfolio is described by the trivariate sum \(X = w_1X_1 + w_2X_2 + w_3X_3\). We assume that the triple \((X_1, X_2, X_3)\) belongs to a trivariate copula with positive dependence of the type (9.7) with gamma marginals. Then the couples \((X_i, X_j)\) are linear Spearman couples with gamma margins and Spearman correlation coefficients \(\rho_{ij}^s\). By the formula (A.21) of Example A.2 one obtains \(\rho_{12}^s = 0.75064, \rho_{13}^s = 0.60239, \rho_{23}^s = 0.50057\). Solving for the dependence parameters \(\theta_{ij}\) in (9.8) one obtains the numerical values \(\theta_{12} = 0.83702, \theta_{13} = 0.70941, \theta_{23} = 0.37137\). Table 10.1 reports some exact and approximate solvency margin calculations for the above trivariate portfolio model, where the solvency margin is defined as in Example 6.2. The exact model (9.7) with gamma margins is compared to the same model with normal margins as well as with the usual trivariate normal model. It is also compared with an approximate model (9.7).
with gamma margins and with a univariate gamma approximation. The approximate model (9.7) supposes that the five trivariate distributions with elementary copulas \( u_1, u_2, u_3, u_4, u_5 \) are replaced by univariate margins with the same means and standard deviations as their trivariate sums. The univariate gamma approximation has the same mean and standard deviation as the trivariate sum of the overall loss. The simple normal models clearly underestimate the solvency margins while the approximate model (9.7) with gamma margins yields reliable measures, which are significantly more accurate than a simple univariate gamma approximation. In view of the computational complexity required for the evaluation of the exact model (9.7), the simpler approximate model (9.7) should be used for practical purposes. Lower and upper bounds, which are obtained under independence and comonotonicity assumptions as in Hürlimann (2001a), are also listed. Again, normal margins underestimate the bounds. Even more, the normal upper bounds underestimate the values for the exact model (9.7) with gamma margins.

**Table 10.1:** Solvency margins for an insurance trivariate portfolio model

<table>
<thead>
<tr>
<th>Model assumptions</th>
<th>Marginal losses</th>
<th>VaR method</th>
<th>CVaR method</th>
</tr>
</thead>
<tbody>
<tr>
<td>Independence</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Trivariate distribution</td>
<td>Normal</td>
<td>16.947</td>
<td>20.144</td>
</tr>
<tr>
<td>Univariate gamma approximation</td>
<td>None</td>
<td>18.936</td>
<td>22.973</td>
</tr>
<tr>
<td>Trivariate distribution</td>
<td>Gamma</td>
<td>20.046</td>
<td>24.660</td>
</tr>
<tr>
<td>Positive dependence</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Usual trivariate normal</td>
<td>Normal</td>
<td>25.296</td>
<td>29.709</td>
</tr>
<tr>
<td>Exact model (9.7)</td>
<td>Normal</td>
<td>26.310</td>
<td>31.507</td>
</tr>
<tr>
<td>Univariate gamma approximation</td>
<td>None</td>
<td>29.066</td>
<td>35.094</td>
</tr>
<tr>
<td>Approximate model (9.7)</td>
<td>Gamma</td>
<td>30.174</td>
<td>37.323</td>
</tr>
<tr>
<td>Exact model (9.7)</td>
<td>Gamma</td>
<td>30.824</td>
<td>39.169</td>
</tr>
<tr>
<td>Comonotonocity</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Trivariate distribution</td>
<td>Normal</td>
<td>29.895</td>
<td>34.978</td>
</tr>
<tr>
<td>Univariate gamma approximation</td>
<td>None</td>
<td>34.882</td>
<td>42.116</td>
</tr>
<tr>
<td>Trivariate distribution</td>
<td>Gamma</td>
<td>35.519</td>
<td>43.064</td>
</tr>
</tbody>
</table>

Diversification effects and risk allocations for the VaR measures according to the covariance and Shapley principles are illustrated in Table 10.2. The values obtained for the exact and approximate models (9.7) with gamma margins are compared.

**Table 10.2:** VaR diversification for an insurance trivariate portfolio model

<table>
<thead>
<tr>
<th></th>
<th>in % risk premium</th>
<th>in absolute value</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>approx.</td>
<td>exact</td>
</tr>
<tr>
<td>Stand-alone values</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
## Appendix: Covariance formulas for the linear Spearman copula.

Starting point is the following result.

**Theorem A.1.** Let \((X, Y)\) be distributed as \(F(x, y) = C_\theta[F_X(x), F_Y(y)]\), where \(C_\theta(u, v)\) is the linear Spearman copula (8.1). Assume that the continuous and strictly increasing marginal distributions are defined on the open supports \((a_X, b_X)\), \((a_Y, b_Y)\). For an arbitrary differentiable function \(\psi(y)\), assume the following regularity assumption holds:

\[
\lim_{y \to a_Y} \psi(y) F_Y(y) = 0,
\]

\[
\lim_{y \to b_Y} \psi(y) \left( E[X | F_Y(y)] - \int_{a_Y}^{b_Y} F_X^{-1}[F_Y(y)] dF_Y(y) \right) = 0. \tag{RA}
\]

Then one has the covariance formula

\begin{table}[h!]
\centering
\begin{tabular}{|l|c|c|c|c|}
\hline
Risk Level & Value & Value & Value & Value \\
\hline
Low risk & 25.3 & 25.3 & 10.14 & 10.14 \\
Moderate risk & 34.9 & 34.9 & 6.98 & 6.98 \\
High risk & 46.0 & 46.0 & 18.41 & 18.41 \\
Sum & 35.5 & 35.5 & 35.53 & 35.53 \\
\hline
\end{tabular}
\end{table}

\begin{table}[h!]
\centering
\begin{tabular}{|l|c|c|c|c|}
\hline
Risk Level & Allocated values & Covariance principle & & \\
\hline
Low risk & 19.7 & 20.1 & 7.87 & 8.04 \\
Moderate risk & 26.7 & 27.3 & 5.34 & 5.46 \\
High risk & 42.4 & 43.3 & 16.96 & 17.32 \\
Portfolio risk & 30.2 & 30.8 & 30.17 & 30.82 \\
\hline
\end{tabular}
\end{table}

\begin{table}[h!]
\centering
\begin{tabular}{|l|c|c|c|c|}
\hline
Risk Level & Diversification effect & & & \\
\hline
Low risk & 5.7 & 5.2 & 2.27 & 2.10 \\
Moderate risk & 8.2 & 7.6 & 1.64 & 1.52 \\
High risk & 3.6 & 2.7 & 1.45 & 1.09 \\
Portfolio risk & 5.4 & 4.7 & 5.36 & 4.71 \\
\hline
\end{tabular}
\end{table}

\begin{table}[h!]
\centering
\begin{tabular}{|l|c|c|c|c|}
\hline
Risk Level & Allocated values & Shapley principle & & \\
\hline
Low risk & 21.8 & 22.4 & 8.71 & 8.96 \\
Moderate risk & 28.3 & 28.1 & 5.67 & 5.63 \\
High risk & 39.5 & 40.6 & 15.79 & 16.23 \\
Portfolio risk & 30.2 & 30.8 & 30.17 & 30.82 \\
\hline
\end{tabular}
\end{table}

\begin{table}[h!]
\centering
\begin{tabular}{|l|c|c|c|c|}
\hline
Risk Level & Diversification effect & & & \\
\hline
Low risk & 3.6 & 2.9 & 1.43 & 1.18 \\
Moderate risk & 6.5 & 6.8 & 1.31 & 1.35 \\
High risk & 6.5 & 5.4 & 2.62 & 2.18 \\
Portfolio risk & 5.4 & 4.7 & 5.36 & 4.71 \\
\hline
\end{tabular}
\end{table}
\[
\text{Cov}[X, \psi(Y)] = \text{sgn}(\theta) \theta \cdot E\left[\left(F_X^{-1}\left[F_Y^\theta(Y)\right] - E[X]\right) \cdot \psi(Y)\right],
\]

(A.1)

where one sets
\[
F_Y^\theta(y) = \begin{cases} F_Y(y), & \theta \geq 0, \\ F_Y^c(y), & \theta < 0, \end{cases}
\]

(A.2)

with \( F_Y(y) = 1 - F_Y(y) \) the survival function.

**Proof.** Let us first derive the regression function \( E[X|Y = y] \). The conditional distribution of \( X \) given \( Y = y \) equals for \( \theta \geq 0 \)
\[
F(x|y) = \frac{\partial C_2}{\partial y} \left[ F_X(x), F_Y(y) \right] = \begin{cases} F_X(x) + \theta F_Y^c(x), & x \geq F_X^{-1}[F_Y(y)], \\ (1-\theta) F_X(x), & x < F_X^{-1}[F_Y(y)], \end{cases}
\]

(A.3)

and for \( \theta < 0 \)
\[
F(x|y) = \begin{cases} (1+\theta) F_X(x), & x < F_X^{-1}[F_Y(y)], \\ F_X(x) - \theta F_Y^c(x), & x \geq F_X^{-1}[F_Y(y)] \end{cases}
\]

(A.4)

Through calculation one obtains the regression formula
\[
E[X|y] = \int_0^y \left[l - F(x|y)\right] dx - \int_{-\infty}^0 F(x|y) dx
\]

(A.5)

\[
= \begin{cases} E[X] - \theta \cdot \left(E[X] - F_X^{-1}[F_Y(y)]\right), & \theta \geq 0, \\ E[X] + \theta \cdot \left(E[X] - F_X^{-1}[F_Y(y)]\right), & \theta \leq 0, \end{cases}
\]

(A.6)

which is a weighted average of the mean \( E[X] \) and the quantile \( F_X^{-1}[F_Y(y)] \) respectively \( F_X^{-1}[F_Y(y)] \). To obtain the stated covariance formula, one uses the well-known formula by Hoeffding(1940) and Lehmann(1966), Lemma 2, to get the expression
\[
\text{Cov}[X, \psi(Y)] = \int_{-\infty}^y \int_{-\infty}^y [F(x, y) - F_X(x) F_Y(y)] \psi'(y) dx dy
\]

(A.7)

Furthermore, from (A.6) one obtains
\[
E[X] - E[X|Y \leq y] = \text{sgn}(\theta) \theta \cdot \left(E[X] - E[F_X^{-1}[F_Y^\theta(Y)] \cdot Y \leq y]\right).
\]

(A.8)

Inserted in (A.7), a partial integration yields
\[ \text{Cov}[X, \psi(Y)] = \text{sgn}(\theta) \cdot \int_{-\infty}^{\infty} \left( E[X]F_y(y) - \int_{-\infty}^{y} F^{-1}_X(F_y(y)) dF_y(y) \right) \psi'(y) dy \]

\[ = \text{sgn}(\theta) \cdot \left\{ \psi(y) \left[ E[X]F_y(y) - \int_{-\infty}^{y} F^{-1}_X(F_y(y)) dF_y(y) \right]_{y}^{y} + \int_{-\infty}^{y} \psi(y) \left( F^{-1}_X(F_y(y)) - E[X]dF_y(y) \right) \right\} \]  

(A.9)

which implies (A.1) by the regularity assumption. \(\diamond\)

The application of this result to margins from a symmetric location-scale family is simple.

**Corollary A.1.** Under the assumptions from Theorem A.1 suppose that

\[ F_X(x) = F_Z\left( \frac{x - \mu_X}{c_X} \right), \quad F_Y(y) = F_Z\left( \frac{y - \mu_Y}{c_Y} \right), \quad \text{with} \quad \mu_X = E[X], \mu_Y = E[Y], \quad \text{and} \quad F_Z(-z) = F_Z(z). \]

Then one has

\[ \text{Cov}[X, \psi(Y)] = \theta \frac{c_X}{c_Y} \text{Cov}[Y, \psi(Y)]. \]  

(A.10)

**Proof.** The result follows from (A.1) noting that \( F^{-1}_X(F^\theta_y(y)) = \mu_X + \text{sgn}(\theta) \frac{c_X}{c_Y} (y - \mu_Y). \) \(\diamond\)

**Remark A.1.** As shown in Section 8.1 one has \( \theta = \rho_s \) the Spearman grade correlation coefficient. If \( \psi(y) = y \) satisfies (RA), and the variances \( \sigma^2_X, \sigma^2_Y \) exist, one has for a symmetric location-scale family

\[ \rho_s \cdot \frac{c_X}{c_Y} = \rho \cdot \frac{\sigma_X}{\sigma_Y}, \]  

(A.11)

where \( \rho \) is Pearson’s correlation coefficient. If the scale parameters are proportional to the standard deviations, that is \( c_X = c \cdot \sigma_X \) and \( c_Y = c \cdot \sigma_Y \) for some \( c \), then \( \rho_s = \rho \). This special case shows why Spearman’s \( \rho_s \) should be interpreted as a measure of correlation. Further special cases under which \( \rho_s = \rho \) will be derived later in this Appendix.

In general, if \( \psi(y) = y \) satisfies (RA), the parameter \( \theta = \rho_s \) remains non-linearly related with Pearson’s correlation coefficient \( \rho \), as the examples below illustrate. If one has to fit the model to data, the estimation of \( \theta = \rho_s \) and \( \rho \) is a very important issue. In case \( \theta > 0 \) the model captures a non-trivial asymptotic dependence by (8.10), and is thus capable to model in some way bivariate extreme values. In the latter situation, the standard product-moment correlation estimator for \( \rho \) has a very bad performance, and there is a need for more robust estimators (e.g. Lindskog(2000)). In this respect the relationship (A.1) opens the way for alternative estimators. For example, if one follows the method of inference function for margins or IFM method studied in McLeish and Small(1988), Xu(1996), and Joe(1997), Section 10.1, one proceeds by doing an estimation of the parameters from the univariate marginal distributions (through separate maximum likelihood estimations) followed by an
estimation of the dependence parameter $\theta$. Inserting the estimated parameters in an explicit analytical expression for (A.1) yields an alternative estimator of the linear correlation coefficient by setting

$$\hat{\rho} = \text{sgn}(\hat{\theta}) \hat{\theta} \cdot \frac{\overline{E[Q_X F_{\theta}(Y) - E[X] \cdot Y]} - E[X] \cdot Y}{\sigma_X \sigma_Y},$$

(A.12)

where the hat indicates insertion of estimated parameters. A new simple estimator for the dependence parameter $\theta = \rho_S$ is provided in the following result.

**Theorem A.2.** Let $\{X_k, Y_k\}$ be a sample of size $N$ from $LS_{\theta}(X, Y)$, $\theta \in [-1,1]$, and let $X_{(k)}, Y_{(k)}$ denote the order statistics taken in the decreasing order. Denote by $irX_{(k)}$ the inverse rank of the observation $X_{(k)}$, defined as the index value $j$ in the original sample such that the rank of $X_j$ in the order statistics is $k$. Then the formulas

$$\hat{\theta} = \frac{\sum_{k=1}^{N} (X_k - \overline{X}) \cdot (Y_k - \overline{Y})}{\sum_{k=1}^{N} (X_{irX_{(k)}} - \overline{X}) \cdot (Y_{irY_{(k)}} - \overline{Y})},$$

(A.13)

$$\text{sgn}(\hat{\theta}) = \text{sgn}\left\{\sum_{k=1}^{N} (X_k - \overline{X}) \cdot (Y_k - \overline{Y})\right\},$$

(A.14)

yield an estimator of Spearman’s coefficient $\theta = \rho_S$.

**Proof.** If $\psi(y) = y$ satisfies (RA), one knows by Theorem A.1 that

$$\text{sgn}(\theta) = \frac{\text{Cov}[X, Y]}{\overline{E[Q_X F_{\theta}(Y) - E[X] \cdot Y]}}. $$

An estimator of the numerator is the usual product moment estimator. For the denominator use that $F_{\theta}(Y) \sim U(0,1)$ to get the symmetric expression

$$E[Q_X F_{\theta}(Y) - E[X] \cdot Y] = E[(Q_X(U) - E[X]) \cdot (Q_{\theta}(U) - E[Y])].$$

To obtain an estimator for this quantity, let us use the empirical quantile functions $Q_X(u), Q_{\theta}(u)$ of the samples. Using Embrechts et al. (1997), p.183, one has the order statistics representation $Q_X(u) = X_{(k)}$ provided $1 - \frac{k}{n} \leq u < 1 - \frac{k-1}{n}$. Equivalently, in terms of the inverse rank function, one has $Q_{\theta}(u) = Y_{irY_{(u)}}$, where $[x]$ denotes the integer part of $x$. The desired estimator is the empirical counterpart of the above symmetric expression. ◊

In the remaining part of the Appendix, several examples illustrate the analytical evaluation of (A.1), which provide explicit relationships between the parameters $\theta = \rho_S$ and $\rho$. 
**Example A.1**: lognormal margins

Suppose that $F_X(x) = D\left(\frac{\ln(x) - \alpha}{\beta_x}\right)$, $F_Y(y) = D\left(\frac{\ln(y) - \alpha}{\beta_y}\right)$, with $D(-z) = D(z)$ (note that if $\theta = \rho < 0$ the function $D(z)$ may be non-symmetric). Then one has

$$Q_x[F_x(y)] = \exp[\alpha_y \cdot \ln(y) + \beta_y], \quad \alpha_y = \text{sgn}(\theta) \cdot \frac{\beta_x}{\beta_y}, \quad \beta_y = \alpha_y - \alpha_x \alpha_y,$$

and under the regularity assumption (RA), one has

$$\text{Cov}[X, \psi(Y)] = \text{sgn}(\theta) \cdot E\left[\exp(\beta_y \cdot Y^{\alpha_y} - E[X]) \cdot \psi(Y)\right]. \quad (A.15)$$

In the special case $D(z) = \Phi(z)$ of lognormal margins, and $\psi(y) = y$, standard calculations show that (RA) is fulfilled and one obtains

$$\text{Cov}[X, Y] = \exp(\theta) \cdot \mu_x \mu_y \cdot \left(\exp(\theta) \cdot \beta_x \beta_y - 1\right), \quad (A.16)$$

where $\mu_x = \exp(\alpha_x + \frac{1}{2} \beta_x^2)$, $\mu_y = \exp(\alpha_y + \frac{1}{2} \beta_y^2)$ denote the means of $X$ and $Y$. In the special case of equal coefficients of variation $k_x = k_y = \sqrt{\exp(\beta_x^2) - 1} = \sqrt{\exp(\beta_y^2) - 1}$ and $\theta \geq 0$, one has

$$\text{Cov}[X, Y] = \theta \cdot \sigma_x \sigma_y, \quad \text{with } \sigma_x, \sigma_y \text{ being the standard deviations. In this special case } \theta = \rho \text{ identifies with Pearson’s } \rho. \text{ In general one has the relationship}$$

$$\rho = \frac{\text{sgn}(\rho_y) \cdot \rho_y \cdot \left(\exp(\theta) \cdot \sigma_x \sigma_y - 1\right)}{k_x k_y}. \quad (A.18)$$

In particular, the extremal bounds for $\rho$, which are attained when $\rho_y = \pm 1$ (e.g. Tchen(1980)), are given by

$$\frac{e^{-\sigma_x \sigma_y} - 1}{k_x k_y} \leq \rho \leq \frac{e^{\sigma_x \sigma_y} - 1}{k_x k_y}. \quad (A.19)$$

This has been originally derived by DeVeaux(1976) as mentioned in Romano and Siegel(1986), Section 4.22. As a significant example, if $\beta_x = 1, \beta_y = 4$, then $\rho$ must be close to zero. This illustrates the fact that by fixed $\rho$ bivariate distributions with lognormal margins do not always exist. In contrast, by fixed $\theta = \rho$, the linear Spearman copula guarantees the existence of a bivariate distribution with lognormal margins.

**Example A.2**: gamma margins
Suppose that $F_X(x) = \Gamma(\beta_X, x, \alpha_X)$, $F_Y(y) = \Gamma(\beta_Y, y, \alpha_Y)$, where $\Gamma(\cdot, \cdot)$ is the Gamma distribution with shape parameter $\alpha$. Restricting the attention to $\theta \geq 0$, one obtains from

$$Q_X[F_Y(y)] = \beta_X^{-1}(\Gamma(\beta_Y, y, \alpha_Y), \alpha_X)$$

and (A.1) the formula

$$\rho = \frac{\theta}{\sigma_X \sigma_Y} \left[ \frac{1}{\beta_X} \int_0^\infty \Gamma^{-1}(\Gamma(\beta_Y, y, \alpha_Y), \alpha_X) y dF_Y(y) - \mu_X \mu_Y \right]$$

If $\alpha_X = \alpha_Y$ (equal coefficients of variation) it is not difficult to show that Spearman’s $\theta = \rho_s$ coincides with Pearson’s $\rho$.

**Example A.3**: Pareto margins

If $X \sim \text{Par}(\lambda_X, \gamma_X)$, $Y \sim \text{Par}(\lambda_Y, \gamma_Y)$ have Pareto survival functions $F_X(x) = (1 + \frac{x}{\lambda_X})^{-\gamma_X}$, $F_Y(y) = (1 + \frac{y}{\lambda_Y})^{-\gamma_Y}$, $x \geq 0$, $\lambda_X, \lambda_Y > 0$, $\gamma_X, \gamma_Y > 1$, one obtains with the quantile function $Q_X[u] = \lambda_X \cdot \left[(1-u)^{-\gamma_X} - 1\right]$ that

$$\text{Cov}[X, Y] = \theta \cdot E \left[ \lambda_X (1 + \frac{Y}{\lambda_Y})^{-\gamma_X} \cdot Y - \lambda_X Y - \mu_X Y \right]$$

Through straightforward calculation one gets

$$I = E \left[ \frac{\gamma_X}{\gamma_X - 1} \right] = \lambda_X \left( \frac{\gamma_X}{\gamma_X - 1} \right) \int_0^\infty \frac{1}{\lambda_Y} \left( (1 + \frac{y}{\lambda_Y})^{-\gamma_Y} - 1 \right) dy$$

(A.23)

Since the last integral represents the mean of a random variable, which is $\text{Par}(\lambda_Y, \frac{\gamma_Y}{\gamma_X} (\gamma_X - 1))$ distributed, one obtains

$$I = \lambda_X \left( \frac{\gamma_X}{\gamma_X - 1} \right) \lambda_Y \left( \frac{\gamma_Y}{\gamma_X \gamma_Y - \gamma_X - \gamma_Y} \right), \quad \gamma_X \gamma_Y > \gamma_X + \gamma_Y.$$  

(A.24)

Inserted in (A.22) one obtains the covariance formula

$$\text{Cov}[X, Y] = \theta \left( \frac{\gamma_X \gamma_Y}{\gamma_X \gamma_Y - \gamma_X - \gamma_Y} \right) \mu_X \mu_Y.$$  

(A.25)
In the special case of equal coefficients of variation $k = \frac{\gamma_x}{\sqrt{\gamma_x - 2}} = \frac{\gamma_y}{\sqrt{\gamma_y - 2}}$, $\gamma_x = \gamma_y > 2$, one has $\text{Cov}[X, Y] = \theta \cdot \sigma_x \sigma_y$, hence $\theta = \rho_s$ coincides with Pearson’s $\rho$ as in the Examples A.1 and A.2.

**Example A.4**: log-double Weibull margins

Let $X \sim \ln DW(\alpha_x, \beta_x, \gamma_x)$, $Y \sim \ln DW(\alpha_y, \beta_y, \gamma_y)$ have log-double Weibull distribution functions

$$F_X(x) = F_{\gamma_x}\left(\frac{\ln(x) - \alpha_x}{\beta_x}\right), \quad F_Y(y) = F_{\gamma_y}\left(\frac{\ln(y) - \alpha_y}{\beta_y}\right),$$

with

$$F_{\gamma_x}(z) = \begin{cases} \frac{1}{2} \exp\left(-\lambda_x \left|z^{\gamma_x}\right|\right), & z \leq 0, \\ 1 - \frac{1}{2} \exp\left(-\lambda_x \left|z^{\gamma_x}\right|\right), & z \geq 0, \end{cases} \quad \lambda_x = \Gamma\left(1 + \frac{2}{\gamma_x}\right), \quad (A.26)$$

$$F_{\gamma_y}(z) = \begin{cases} \frac{1}{2} \exp\left(-\lambda_y \left|z^{\gamma_y}\right|\right), & z \leq 0, \\ 1 - \frac{1}{2} \exp\left(-\lambda_y \left|z^{\gamma_y}\right|\right), & z \geq 0, \end{cases} \quad \lambda_y = \Gamma\left(1 + \frac{2}{\gamma_y}\right). \quad (A.27)$$

A calculation shows that

$$Q_X[F_Y(y)] = \exp\left[\alpha_x + \beta_x \left(\frac{\lambda_y}{\lambda_x}\right)^{\gamma_x} \text{sgn}[\ln(y) - \alpha_y] \ln(y) - \alpha_y \frac{\gamma_x}{\beta_y}\right]. \quad (A.28)$$

Let $f_Y(x) = \frac{1}{\beta_y} f_{\gamma_y}\left(\frac{\ln(x) - \alpha_y}{\beta_y}\right)$ the density of $Y$, where $f_{\gamma_y}(z) = \frac{1}{2} \lambda_y \left|z^{\gamma_y^{-1}}\right| \exp\left(-\lambda_y \left|z^{\gamma_y}\right|\right)$. With the substitution $\ln(y) - \alpha_y = \text{sgn}[\ln(y) - \alpha_y] \beta_y z$ one obtains

$$I = E\left[Q_X[F_Y(y)], Y\right] = e^{\alpha_x + \alpha_y} \cdot (I_1 + I_2), \quad (A.29)$$

$$I_1 = \int_0^\infty \exp\left(-\beta_x \left(\frac{\lambda_y}{\lambda_x}\right)^{\gamma_x} - \beta_y z \right) \frac{1}{2} \gamma_y \lambda_y z^{\gamma_y^{-1}} \exp\left(-\lambda_y z^{\gamma_y}\right) \text{d}z, \quad (A.30)$$

$$I_2 = \int_0^\infty \exp\left(\beta_x \left(\frac{\lambda_y}{\lambda_x}\right)^{\gamma_x} + \beta_y z \right) \frac{1}{2} \gamma_y \lambda_y z^{\gamma_y^{-1}} \exp\left(-\lambda_y z^{\gamma_y}\right) \text{d}z. \quad (A.31)$$

The further substitution $u = \lambda_y z^{\gamma_y}$ yields

$$I_1 = \int_0^\infty \exp\left(-\beta_x \left(\frac{u}{\lambda_x}\right)^{\gamma_x} - \beta_y \left(\frac{u}{\lambda_y}\right)^{\gamma_y}\right) e^{-u} \text{d}u. \quad (A.32)$$

Expanding the first exponential expression in a Taylor series and using the integral definition of the Gamma function, one sees that
\[
I_1 = \frac{1}{2} \sum_{k=0}^{\infty} (-1)^k \left\{ \sum_{j=0}^{k} \left[ \frac{\beta_x}{\lambda_x} \right]^{j} \left[ \frac{\beta_y}{\lambda_y} \right]^{k-j} \Gamma \left( 1 + \frac{j}{\gamma_x} + \frac{k-j}{\gamma_y} \right) \right\}.
\]

Similarly, one obtains
\[
I_2 = \frac{1}{2} \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{j=0}^{k} \left[ \frac{\beta_x}{\lambda_x} \right]^{j} \left[ \frac{\beta_y}{\lambda_y} \right]^{k-j} \Gamma \left( 1 + \frac{j}{\gamma_x} + \frac{k-j}{\gamma_y} \right).
\]

Inserting (A.33) and (A.34) into (A.29) and using (A.1) one obtains the covariance formula
\[
\text{Cov}[X,Y] = \theta \cdot \left( 2 \cdot e^{\alpha_x + \alpha_y} \sum_{m=0}^{\infty} \frac{1}{m!} \left[ \frac{\beta_x}{\lambda_x} \right]^{2m-j} \left[ \frac{\beta_y}{\lambda_y} \right]^{2m-j} \Gamma \left( 1 + \frac{j}{\gamma_x} + \frac{2m-j}{\gamma_y} \right) \right) - \mu_x \mu_y.
\]

Consider the interesting special case of the log-Laplace margins obtained setting \( \gamma_x = \gamma_y = 1 \), for which this simplifies to
\[
\text{Cov}[X,Y] = \theta \cdot \left( \frac{2 \cdot e^{\alpha_x + \alpha_y}}{2 - (\beta_x + \beta_y)} - \mu_x \mu_y \right).
\]

In the special case of equal coefficients of variation, whose squares are given by
\[
k_x^2 = \frac{1}{2} \frac{(2 - \beta_x^2)^2}{2 - 4 \beta_x^2} - 1 = k_y^2 = \frac{1}{2} \frac{(2 - \beta_y^2)^2}{2 - 4 \beta_y^2} - 1, \quad \beta_x = \beta_y < \frac{\sqrt{2}}{2},
\]

one obtains using \( \mu_x = \frac{2}{2 - \beta_x^2} e^{\alpha_x} \), \( \mu_y = \frac{2}{2 - \beta_y^2} e^{\alpha_y} \), that \( \text{Cov}[X,Y] = \theta \cdot \sigma_x \sigma_y \), hence \( \theta = \rho_s \) coincides with Pearson’s \( \rho \) as in the Examples A.1 to A.3. Note that the last property also holds for the more general case \( \gamma_x = \gamma_y > 1 \), whose verification is left to the reader.

\section*{References}

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