A Theoretical Framework for Incorporating Scenarios into Operational Risk Modeling*

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Abstract In this paper, I introduce a theoretically justified framework that incorporates scenario analysis into operational risk modeling. The basis for the framework is the idea that only worst-case scenarios contain valuable information about the tail behavior of operational losses. In addition, worst-case scenarios introduce a natural order among scenarios that makes possible a comparison of the ordered scenario losses with the corresponding quantiles of the severity distribution that research derives from historical losses. Worst-case scenarios contain information that enters the quantification process in the form of lower bound constraints on the specific quantiles of the severity distribution. The framework gives rise to several alternative approaches to incorporating scenarios.

Keywords Operational risk · Scenario analysis · Constrained estimation · The Markov chain Monte Carlo method · Stochastic dominance

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1 Introduction

Although operational risk is a relatively new discipline, it has already gained international recognition due to Basel II (2005), which is a series of recommendations regarding how large financial institutions should measure and manage their risks to further strengthen

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the soundness and stability of the international banking system. Accordingly, Basel II recommends that large financial institutions hold enough capital to cover their operational risk, which it defines as the risk of loss resulting from inadequate or failed internal processes, people and systems, or from external events. This definition includes legal risk, but excludes strategic and reputational risk. Because the sources of operational risk are numerous, Basel II allows banks to model operational risk separately in each risk cell, which is also called a "unit of measure."

Being able to accurately estimate the amount of capital that financial institutions need to hold against operational risk is a key challenge to satisfying Basel II requirements. Because financial institutions only began collecting operational risk data recently, information from historically observed data is often insufficient to model operational risk reliably. A need exists for additional sources of information such as scenarios-hypothetical realizations of an institution's, and broadly speaking the financial industry's, inherent risks. Therefore, the Final Rule (2007) – the adaptation of Basel II by the U.S. national supervisory authorities through domestic rule-making procedures – requires that large, internationally active financial institutions incorporate scenario analysis into their operational risk assessment and quantification systems. The underlying assumption is that scenarios contain important information about severe but plausible future losses that have not yet been seen in historical observations. The incorporation of scenario analysis should bring that information to the quantification process and lead to a more accurate capital assessment process.

Unfortunately, the Final Rule does not provide any specifics on how the financial institutions should incorporate this valuable source of information. So far, limited success exists in meeting this requirement. Not surprisingly, an important aspect of the current practice is the absence of widely accepted and theoretically well-grounded approaches to incorporating scenarios. Under these circumstances, practitioners use a variety of ad-hoc approaches. For example, in one extreme an institution might simply ignore its scenarios by stating that the level of judgment, which it uses in the process of formulating scenarios, is too high and the loss estimates are too unreliable for them to be comfortable with the results. In the other extreme, an institution might decide to discard its own historical losses and come up with a capital estimate that comes solely from its scenarios. A common feature of all these approaches is the lack of a sound theoretical justification behind them. Dutta and Babbel (2010) attempt to resolve the challenge of incorporating scenarios by proposing a simulation-based approach. Although their approach seems reasonable, the authors do not fully disclose its theoretical underpinnings. As a result, the reader has difficulties understanding the theoretical justifications behind their approach. It should be noted that Berkowitz (2000) also studies a simulation-based approach to incorporating scenarios, but in the context of stress testing market-risk models.

In this paper, I introduce a framework that links scenarios to historical losses and has a basis in solid theoretical considerations. This framework derives from the concept that the focus should be on worst-case scenarios only, because only these scenarios contain valuable information about the tail behavior of operational losses. In addition, worst-case scenarios introduce a natural order among scenarios and provide the possibility for a comparison between ordered scenario losses and the corresponding quantiles of the severity distribution that historical losses determine. The information from worst-case scenarios enters the quantification process in the form of lower bound constraints on the specific quantiles of the severity distribution. This framework gives rise to several alternative approaches to the incorporation of scenarios.

The introduction of ordering among scenarios also addresses one important challenge of the scenario generation process. Namely, experts can generate too many scenarios in certain loss regions and too few scenarios in some other regions. As a result, scenario-implied frequencies of losses in different regions might not necessarily follow true loss frequencies. Alternatively, they might generate too many scenarios overall so that scenario frequencies become inflated. The main source of the challenge is that no knowledge exists on true loss frequencies. One way of dealing with this challenge is to adjust scenario frequencies by some multiplier to align them with historical loss frequencies. However, such an adjustment is questionable because it implies that true frequencies are those of backward-looking historical losses, which might not necessarily be the case. Dutta and Babbel (2010) propose the replacement of historical frequencies with scenario frequencies whenever the former are lower than the latter. Although this assumption seems conservative, if too many scenarios have been generated from the body relative to the tail of the severity distribution, then this situation might lead to a reduction in capital.[1] Therefore, whether the reduction in

[1]E.g., if line of business managers are responsible for generating scenarios, they might avoid generating

capital happens due to expert opinions, or simply because of a disproportionate amount of scenarios, is not clear.[2] Alternatively, if scenario experts generate too many scenarios from the tail, then one might observe an increase in capital. Moreover, a danger exists of capital becoming a strictly increasing function of the number of tail scenarios. The framework that I am proposing practically eliminates this challenge by shifting the focus to worst-case scenarios. Adding more scenarios does not affect the set of worst-case scenarios as long as for each newly generated scenario, there exists a worst-case scenario with a lower frequency and a higher severity.

The main purpose of this paper is to build a theoretically justified framework for incorporating scenarios for the purpose of estimating scenario-adjusted operational risk capital. The paper does not address several other important challenges of incorporating scenarios. For example, I do not account for the well known biases of the scenario generation process, which I briefly discuss in Section 3. Also, the paper does not account for the possibility of multiple scenarios with the same underlying hypothetical event being dependent. Instead, I assume that scenarios that belong to the same unit of measure are independent, which is simply the extension of a standard assumption about observed losses.[3] Lastly, this paper greatly simplifies the process of extracting conditional information from scenarios. Instead of assigning probability distributions to scenarios, the paper uses an approach that elicits only a lower bound of the worst-case loss amount conditioned on the occurrence of a scenario. Benefits and drawbacks exist to assigning probability distributions to scenarios. On the benefit side, the extraction of more information might occur from expert knowledge about the distribution of a scenario than from just a lower bound of the worst-case loss amount. Also, when scenarios are presented with their probability distributions, it is easier to incorporate them into the quantification framework. The main drawback is that scenario experts generally have very limited knowledge about probabilistic concepts. Therefore, the

large scenario losses if doing so will negatively affect their performance. I thank Jose Fillat for pointing out this possibility.

[2]E.g., suppose the modeler samples 20,000 losses from the base model and adds 1000 scenario losses to augment the original sample. The 99.9th percentile loss of the original sample, say X, is the 20th largest loss of that sample, while the 99.9th percentile of the augmented sample with 21,000 losses is the 21th largest loss. This difference means that if all 1000 scenario losses are less than X, then the new 99.9th percentile is less than X as well. This situation can lead to lower, scenario-augmented capital if the severity distribution is a heavy-tail, which the single loss approximation shows (Böcker and Klüppelberg 2005)

[3]Nevertheless, the dependence between scenarios belonging to different units of measure can still be captured through the introduction of a dependence structure between different units of measure, say, by means of copulas (see, among others, El Gamal, Inanoglu, and Stengel 2007).

task of assigning probability distributions around scenarios requires a sophisticated process of eliciting expert opinion: someone with a good knowledge of statistics, i.e., an elicitor who needs to ask the "right" questions and then transform the experts' answers into probability distributions. This process brings additional subjectivity to the process of incorporating scenarios. For example, the elicitor might elicit expert opinion about a few quantiles (such as, the median, the 90th, and the 99th percentile quantiles) of the potential loss resulting from a particular scenario and try to fit a probability distribution to those data points. But, the results can become sensitive to the chosen set of quantiles, distribution of losses, and fitting method. Also, the assumption that practitioners possess the required expertise to be able to conduct a sophisticated elicitation process is overly optimistic. Under these circumstances, the preference is to take a minimalistic approach by eliciting a minimum amount of expert information, the formulation of which is in very simple terms and does not require additional processing. This is the approach I take in this paper with regard to scenario elicitation. The observation that the conditional distributions of the worst-case scenarios are truncated from below is the basis for this approach.[4] Therefore, to elicit expert opinion on lower bounds for scenario loss amounts makes sense.

The organization of the paper is as follows. Section 2 briefly describes the current practice of building and interpreting scenarios. Section 3 discusses some of the available ad-hoc approaches to incorporating scenarios and their weaknesses. Section 4 develops the theoretical framework of the paper. Section 5 presents several alternative approaches to the incorporation of scenarios that come from the theoretical framework of Section 4. Section 6 concludes.

2 Interpreting scenarios

In this section, I discuss the current range of practice in building and interpreting scenarios. The basis for the discussion is the so called loss distribution approach (LDA) to modeling operational risk, which is arguably becoming the most popular approach among practitioners. The LDA separately models the severity and the frequency distributions of losses and finds the annual total loss distribution through the convolution of the two distributions.

[4]E.g., if the loss amount of the worst-in-a-two-year scenario is L, then the loss amount of, say, the worst-in-a-three-year scenario cannot be less than L.

Because the total loss distribution is not always analytically tractable, one usually calculates the regulatory capital as the 99.9th percentile of the total loss distribution through Monte-Carlo simulations.

Scenarios are hypothetical realizations of an institution's, or broadly speaking the financial industry's, inherent risks. Therefore, they can be useful in making forward looking adjustments to the frequency and severity of comparable historical events to properly account for the latest or anticipated future changes in the institution's risk profile and business environment that historical losses do not yet reflect. Scenario experts build each scenario as a hypothetical realization of a specific risk under specific circumstances. Nevertheless, some consensus has emerged among scenario experts that each scenario needs to be assigned a duration: the number of years during which this particular scenario happens only once on average. Following this consensus, I call a scenario a "once-in-a-M-year" scenario if it has a duration of M years. When assigning loss severities to scenarios, the range of practice varies substantially. Some institutions assign loss ranges with lower and upper bounds, while other institutions assign just lower bounds, or point estimates of the anticipated loss amounts. This situation creates a number of challenges for the process of incorporating scenarios. First, the duration and the loss range (or the loss amount) do not fully identify scenarios, because the experts can assign the same duration and/or loss range to different scenarios. Second, some scenarios with large durations still can be assigned low loss ranges and vise versa. Third, scenarios with extremely large durations, such as a "once-in-a-thousand-year" scenario, can be in practically any loss range, including the body of the severity distribution, as long as the loss range is small enough to make that scenario a rare hypothetical event. Further, the number of scenarios can be substantial. Therefore, a proper treatment of the available scenario information is necessary to avoid being overwhelmed with the abundance of, sometimes practically useless, information.

Although each expert-generated scenario is important from the risk management perspective, I argue that from the risk quantification perspective the focus should be on worst-case scenarios only. I call these scenarios the "worst-in-a-certain-year" scenarios to emphasize their duration. The worst-in-a-M-year event (where M is a natural number) is a rare event that results in the largest loss the institution experiences once in every M years, on average. The reader needs to keep in mind that scenarios are forward looking statements about some hypothetical events. Therefore, throughout the paper, I use the words scenario and event interchangeably. Because the worst-in-a-M_1-year loss must be less severe than the worst-in-a-M_2-year loss as long as $M_1 < M_2$, the worst-in-a-certain-year events introduce a natural order to scenario losses and make possible the comparison of these losses with the corresponding quantiles of the base model's severity distribution. Throughout the paper, the base model refers to a model that one uses to fit the historical operational losses. In Subsection 4.2, I propose a concise rule for identifying the worst-in-a-certain-year scenarios from the pool of all scenarios.

The formal definition of the "worst-in-a-certain-year" event, which is given in Subsection 4.1, indicates that from the quantification perspective the most efficient way of presenting scenarios is to assume that a duration and a lower bound accompanies each scenario. Such a representation of scenarios for the purpose of risk quantification accommodates all of the above mentioned representations of scenario loss severities. For example, if a scenario comes with a range, one can choose the lower bound of that range as the scenario's lower bound for the quantification purpose. If only the point estimates of scenario losses are available, one can still conservatively use those point estimates as the lower bounds.[5] Importantly, this proposed framework can still be used in situations where the definition of scenarios involves probability distributions, with the provision that each scenario distribution possesses a unique and strictly positive lower bound.

3 Challenges of incorporating scenarios

Unfortunately, scenarios come with their own biases, such as anchoring, confirmation, availability, and overconfidence. For example, anchoring is a common human tendency to rely too heavily, or "anchor," on one trait or piece of information when making decisions. Tversky and Kahneman (1974) introduced the concept that usually once the anchor is set, a bias exists toward that value. For descriptions of the other biases see Kahneman, Slovic, and Tversky (1982). Although each of these biases can have significant capital implications, I do not discuss scenario biases in detail because the main focus of this paper is on resolving the quantitative challenges around the incorporation of scenarios. Throughout the paper

[5] The approach that Dutta and Babbel (2010) propose works with scenario ranges only. If scenario experts report only point estimates of scenario losses, then one has to create a range around them. The authors recommend a 20% range, which makes their approach sensitive to this number.

I assume that scenarios are the result of the best efforts of scenario experts, who understand the business and have sufficient knowledge about the scenario analysis process and its biases. Therefore, I accept the given scenarios without questioning the expert judgment behind them even if the scenarios are quite different from historical experiences. Instead, I assume that, when scenarios are different from historical experiences, they must be conveying important information about possible changes in the institution's risk profile and/or business environment that need to be properly incorporated into the quantification system. Section 5 contains a brief discussion of the effect of scenario biases on the scenario-driven capital results.

Current practice relies on a wide range of ad-hoc approaches for incorporating scenarios. For example, in one extreme an institution might simply ignore its scenarios by stating that the level of judgment in the process of formulating scenarios is too high and the loss estimates are too unreliable for them to be comfortable with the results. In the other extreme, an institution might decide to discard its own historical losses and come up with a capital estimate that comes solely from its scenarios. Approaches exist that lie in between these two extremes. A common feature of all these approaches is the lack of a sound theoretical justification behind them.

To give specific examples, modelers might simply pool scenario losses together with historical loss observations to calculate scenario-driven capital numbers. As Dutta and Babbel (2010) rightly point out, pooling does not take into account scenario frequencies. Because pooling substantially overstates the scenario weights (especially those of the most severe scenarios), an overstatement of scenario-driven capital numbers is most likely to exist as well. Under such circumstances modelers might decide to treat the largest scenarios as being unrealistic even though this might not necessarily be the case. Alternatively, modelers might treat scenario losses as add-ons to capital. Although this approach is among the most conservative, its use by modelers can artificially inflate scenario frequencies that might again lead to the rejection of capital numbers that are unrealistically large. Therefore, both pooling and using scenarios as add-ons should be avoided. Modelers also might decide to use scenario information to choose the candidate severity distribution whose tail shape, through extrapolation, most closely resembles the tail behavior that scenarios imply. Although this approach seems reasonable, it is not clear how to advance if scenario frequencies do not align with observed loss frequencies. For example, no solid reasons exist for adjusting scenario frequencies downward by a scaling factor under the assumption that scenario frequencies overstate true frequencies. Such an adjustment is questionable because it implies that true frequencies are those of the backward-looking historical losses, which might not necessarily be the case.

Alternatively, modelers might treat scenarios as direct comparatives to capital and choose the severity distribution that yields a capital number more in-line with the most severe scenario loss. This treatment might make sense only under heavy-tailed severity distributions as the single loss approximation formula indicates (Böcker and Klüppelberg 2005). Even in that case, the largest scenario's percentile is different from the 99.9th percentile of the total loss distribution. Suppose modelers are aware of this mismatch and they make corrections to the scenario quantile using the single loss approximation. Even after this rightful correction, the approach accommodates the incorporation of a specific scenario that corresponds to a specific percentile of the severity distribution. In addition, using the single loss approximation for this purpose might not work well because its accuracy is questionable unless the severity distribution's tail is very heavy and the percentile is very high (Sahay, Wan, and Keller 2007).

Modelers also might decide to simulate scenarios from their empirical distribution and add them to the pool of historical losses. This process leads to a combined severity distribution, which will be used to calculate scenario-adjusted capital. Berkowitz (2000) first proposes such an approach for incorporating scenarios in the context of measuring market risk. However, Berkowitz's approach is not directly applicable to operational risk modeling mainly because of the necessity to separately model the severity and frequency of losses. Dutta and Babbel (2010) propose a similar simulation-based approach that takes into account the specifics of modeling operational risk. However, the authors do not fully disclose a theoretical justification for their approach. As a result, several important questions still remain unanswered.

4 A theoretical framework for incorporating scenarios

The knowledge gap about the true loss distribution creates a challenging situation. Two pieces of potentially conflicting information (both being imperfect and noisy) exist about the true loss distribution. The first piece of information comes from historically observed losses as the base model, while the other one comes from the expert-generated hypothetical scenarios. The challenge manifests itself in the observation that no prior knowledge exists that would allow for a strict preference for one over the other, or some combination of both, because no knowledge exists on what the true distribution looks like. Therefore, in this kind of situation, the preference is to pursue an approach that processes these two pieces of information and relies on conservatism as an extra layer of protection against potential dangers of the above mentioned knowledge gap. This built-in conservatism of the proposed framework protects the modeler from underestimating capital by allowing the modeler to incorporate the scenario information, at a given quantile level, only when it is in discordance with the base model's severity distribution and to discard the scenario information, at a given quantile, if it is in concordance with the base model's severity distribution. Subsection 4.3 provides the definitions of concordance and discordance below. The scenario constraints that I derive in the same section help identify the discordant worst-case scenarios.

4.1 The worst-in-a-certain-year event and its properties

To show how the worst-in-a-certain-year events can be useful in incorporating scenarios, I start with a formal definition of these events that uses standard probabilistic concepts, and I study their properties. For the sake of simplicity, I assume that all cumulative distribution functions (cdf) that the paper considers are continuous and strictly monotonic to well define their inverse functions. This assumption can be relaxed at the expense of an additional notation but the relaxation adds only marginal value to the results. First, I present two seemingly different definitions of the worst-in-a-certain-year event capturing the reasoning behind this event that Section 2 discusses. Then I show that both definitions are equivalent.

Definition 1 *For any natural number M the definition of the loss amount of the worst-in-a-M-year event, V, is*

$$P\Big\{max(X_1, ..., X_n) > V\Big\} = \frac{1}{M}, \tag{1}$$

where $X_1, ..., X_n$, are observed losses, and n is a random variable representing the annual frequency of losses.

A reasonable explanation for the above definition is as follows. According to the law of large numbers, for a large M, an event with probability p occurs, on average, Mp times. Because $p = 1/M$ in Definition 1, on average the maximum exceeds V only once in M years, which makes this event the worst-in-a-M-year event.

Definition 2 *For any natural number M, the definition of the loss amount of the worst-in-a-M-year event, U, is*

$$P\Big\{\sum_{i=1}^{n} I_{\{X_i>U\}} \geq 1\Big\} = \frac{1}{M}, \tag{2}$$

where $X_1, ..., X_n$, are observed losses, n is a random variable representing the annual frequency of losses, and I_A denotes the indicator function of event A.

Definition 2 defines the worst-in-a-M-year event loss, U, as the amount at which the probability that an annual loss sample contains at least one loss exceeding U is $1/M$.

Proposition 1 *If the probability distribution of losses, P, is continuous then Definitions 1 and 2 of the worst-in-a-M-year loss are equivalent, i.e. $V = U$ for any given natural M.*

Proof Suppose U is such that (2) holds. Then

$$\begin{aligned} P\Big\{\sum_{i=1}^{n} I_{\{X_i>U\}} \geq 1\Big\} &= 1 - P\Big\{\sum_{i=1}^{n} I_{\{X_i>U\}} = 0\Big\} = 1 - P\Big\{X_1 \leq U, ..., X_n \leq U\Big\} \\ &= 1 - P\Big\{max(X_1, ..., X_n) \leq U\Big\} = P\Big\{max(X_1, ..., X_n) > U\Big\}. \end{aligned}$$

Thus, $P\Big\{max(X_1, ..., X_n) > U\Big\} = \frac{1}{M}$. By comparing the last equation with (1) and taking into account the continuity of P, then $U = V$.

The worst-in-a-certain-year loss severity must depend on the institution's annual loss frequency. To understand this reasoning, imagine two institutions with the same severity distribution but different average annual frequencies. If the average annual frequency of the first institution is greater than that of the second institution, then the worst-in-a-certain-year loss of the first institution must be greater than that of the second institution. The next proposition formalizes this reasoning by describing the exact relation between the distribution of the worst-in-a-certain-year loss and the severity distribution.

Proposition 2 *Suppose the losses $X_1, X_2, ...,$ are independent and identically distributed according to the cdf $F(\cdot)$ and the annual frequency of losses, n, is Poisson with the mean parameter $\lambda > 0$. Denote by $G(\cdot)$ the cdf of the maximum loss:*

$$G(v) = P\big\{max(X_1, ..., X_n) \leq v\big\}, v \geq 0.$$

Then the following identity holds.

$$G(v) = \exp\{-\lambda[1 - F(v)]\}, v \geq 0. \tag{3}$$

Proof This is a well known result (see, for example, Böcker and Klüppelberg (2005)).

Equation (3) is the key relation in incorporating information contained in the worst-in-a-certain-year scenarios into the severity distribution of the base model. In Subsection 4.3, I use this relation to derive the scenario constraints that I impose on the base model's severity distribution. Because the scenario constraints involve the worst-in-a-certain-year scenarios, one needs some rule to identify those scenarios from the pool of all scenarios. The description for this rule is in the next section. The last result of this section, Proposition 3, creates the theoretical base for the rule. It is reasonable that among the two worst-in-a-certain-year events, the one with a larger duration must be more severe than the other. The following proposition formalizes this reasoning by introducing a logical order among the worst-in-a-certain-year scenarios.

Proposition 3 *If $M_1 < M_2$, then the loss amount of the worst-in-a-M_1-year event is always less than that of the worst-in-a-M_2-year event.*

Proof Proposition 3 is a direct consequence of (3) and the assumption that the cdf $F(v)$ is a strictly monotonic function, which makes the cdf of the maximum loss, $G(v)$, strictly monotonic as well.

Proposition 3 shows that the lower bounds of the worst-case scenarios do not overlap by construction except for a special situation in which two distinct worst-case scenarios have the same lower bound and the same duration. However, this situation can be remedied. If $S_1^w = (L, M)$ and $S_2^w = (L, M)$ are two such scenarios, then they can be combined into one worst-case scenario $S^w = (L, M/2)$, under the assumption that the two scenarios occur independently.

4.2 Identifying the worst-in-a-certain-year scenarios

As mentioned earlier, for risk quantification purposes I assume that the summary of each scenario comprises both a duration and a lower bound. More formally, I denote the set of all scenarios as $\mathbf{S} = (S_1, ..., S_k)$, where $S_i = (M_i, L_i)$, M_i is a natural number indicating that scenario S_i is a once-in-a-M_i-year event, $L_i > 0$ is the lower bound of the loss associated with this scenario, and k is the total number of scenarios. Because k is usually a large number, the need exists to identify scenarios that are the most useful from the risk quantification perspective and to focus only on them. Proposition 3 establishes worst-in-a-certain-year scenarios that possess a very useful property. Namely, the worst-case scenarios introduce a natural order among scenarios that make a comparison possible between the ordered scenario losses and the corresponding quantiles of the base model's severity distribution. The following rule identifies the set of worst-in-a-certain-year scenarios, $\mathbf{S}^w$, within the set of all scenarios $\mathbf{S}$.

A simple rule for identifying the worst-in-a-certain-year scenarios

Step 1 Find in $\mathbf{S}$ the scenario with the largest lower bound and denote it by $\hat{S}$ for convenience; remove $\hat{S}$ from $\mathbf{S}$ and add it to $\mathbf{S}^w$.

Step 2 Throw away all scenarios in $\mathbf{S}$ with the durations that are equal to or greater than the duration of $\hat{S}$.

Step 3 Repeat Steps 1 and 2 until $\mathbf{S}$ is empty.

A clear reason exists for throwing away scenarios with durations that are greater than or equal to the duration of $\hat{S}$. Among those scenarios no informative scenarios exist, i.e. scenarios with lower bounds that are higher than the lower bound of $\hat{S}$.

4.3 The scenario constraints

In this subsection, I distinguish two types of probability distributions: those of the base model with a subindex of b and the scenario probability distributions with a subindex of s. I assume that scenarios have a common unobserved continuous distribution and each scenario is a random realization from that distribution with unknown loss amount and duration. The scenario experts only need to assign a duration and a lower bound of the unknown loss amount for each scenario to be able to implement the proposed framework. Of course drawbacks and benefits exist for this assumption. The main drawback is that the proposed framework does not readily accommodate the situation where a probability distribution defines each scenario, except for the case when individual scenario distributions have unique positive lower bounds. The most important benefit is that this assumption does not require experts to think in terms of probabilities, because scenario experts are usually not familiar with probabilistic concepts. Instead, experts just need to come up with a reasonable lower bound for a particular scenario and its estimated duration. These two are familiar concepts for scenario experts.

Suppose applying the filter of Subsection 4.2 to $\mathbf{S}$ leads to a set $\mathbf{S}^w$ of the worst-in-a-certain-year scenarios that contains r scenarios, where $1 \leq r \leq k$. Now, Definition 1 and Proposition 2 show that scenario information can be brought to the base model through the comparison of the sample distribution of $\mathbf{S}^w$, which I denote by $G_s(\cdot)$, and the maximum loss distribution of historical losses, $G_b(\cdot)$. Unfortunately, not much is known about $G_s(\cdot)$ except for the information that for the unknown loss V_i of scenario $S_i^w = (M_i^w, L_i^w)$,

$$V_i > L_i^w \text{ and } G_s(V_i) = 1 - \frac{1}{M_i^w},$$

which implies the following inequalities:

$$G_s^{-1}\Big(1 - \frac{1}{M_i^w}\Big) > L_i^w,\ i = 1, ..., r.$$

Definition 3 *A scenario $S^w = (M^w, L^w)$ is concordant with the base model if*

$$G_b^{-1}\Big(1 - \frac{1}{M^w}\Big) > L^w. \tag{4}$$

Inequality (4) means that the historical maximum loss distribution intersects the $1 - \frac{1}{M}$-th quantile at some loss amount that is greater than L^w, so does the scenario loss distribution $G_s(\cdot)$ by its definition.

Proposition 4 *If a scenario $S^w = (M^w, L^w) \in \mathbf{S}^w$ is concordant with the base model where M^w is a natural number greater than one, then*

$$F_b^{-1}\left(1 + \frac{1}{\lambda}\log\Big(1 - \frac{1}{M^w}\Big)\right) > L^w, \tag{5}$$

where $F_b(\cdot)$ is the cdf of the base model's severity distribution.

Proof Inequality (5) is derived from (4) and (3).

Proposition 4 is the main result of the paper. It describes the relation between the base model's severity distribution and the scenarios $\mathbf{S}^w$ at specific quantiles. Scenarios that are concordant with the base model are uninformative in the sense that no apparent reason exists for making any adjustments to the base model's severity distribution at the corresponding quantiles. In contrast, scenarios that are discordant with the base model are informative because they signal a possible shift in the institution's risk profile as well as its true loss distribution. If scenario $S^w = (M^w, L^w)$ is discordant with the base model, then $F_b^{-1}\Big(1 + \frac{1}{\lambda}\log(1 - 1/M^w)\Big) \leq L^w$, which implies that the scenarios are leading to a more conservative estimate of the unknown true loss at the $1 - 1/M^w$-th quantile than the base model's severity distribution.

4.4 Some implications of the scenario constraints

The following table demonstrates that the scenario constraints (5) are almost always imposed at very high quantile levels of the base models's severity distribution. Even under

M	2	5	10	20	50	70	100
$\lambda = 10$	0.9307	0.9777	0.9895	0.9949	0.9980	0.9986	0.9990
$\lambda = 20$	0.9653	0.9888	0.9947	0.9974	0.9990	0.9993	0.9995
$\lambda = 50$	0.9861	0.9955	0.9979	0.9990	0.9996	0.9997	0.9998
$\lambda = 100$	0.9931	0.9978	0.9989	0.9995	0.9998	0.9999	0.9999
$\lambda = 200$	0.9965	0.9989	0.9995	0.9997	0.9999	0.9999	0.9999
$\lambda = 500$	0.9986	0.9996	0.9998	0.9999	1.0000	1.0000	1.0000

Table 1 The quantile levels at which the scenario constraints are imposed on the base model's severity distribution. The quantiles are computed using the formula $q = 1 + \frac{1}{\lambda}\log(1 - \frac{1}{M})$.

the assumption that the average annual frequency of losses in a particular unit of measure is only 10, the worst-in-a-two-year scenario constraint is imposed at the 0.931st quantile level of the severity distribution, while the worst-in-a-hundred-year scenario constraint is imposed in the 0.998th quantile level. Also, the quantile levels of the severity distribution corresponding to the worst-case scenarios with the durations of 100 years and beyond exceed the 0.999th quantile level. Obviously, the accuracy of both the severity as well as the worst-case scenario losses reduces dramatically beyond the quantile levels corresponding to the duration of 100 years. Therefore, Table 1 implies that for the purpose of incorporating scenarios, scenario experts should keep the duration of the worst-case scenarios below 100 years.

One might think that the lower bound $F_b^{-1}\Big(1 + \frac{1}{\lambda}\log(1 - 1/M^w)\Big)$ in (5) can be fairly accurately approximated using the first order Taylor approximation of $q_1 \cong 1 - \frac{1}{\lambda M}$ of $q = 1 + \frac{1}{\lambda}\log(1 - \frac{1}{M})$. The attractiveness of this approximation, if it is accurate, stems from the observation that the inequality (5) can then be replaced by the following much simpler inequality $F_b^{-1}(q_1) > L^w$, which is equivalent to

$$1 - F_b(L^w) > \frac{1}{\lambda M}. \quad (6)$$

However, the following calculations show that the approximation (6) is sometimes quite inaccurate and needs to be avoided. Obviously, it is always true that $q < q_1$; therefore, $F_b^{-1}(q) < F_b^{-1}(q_1)$. Now, consider scenarios in which the lower bounds L^w satisfy the inequalities $F_b^{-1}(q) < L^w < F_b^{-1}(q_1)$. These scenarios are discordant with the base model, but the approximation (6) wrongly implies that they are concordant with the base model. To give an example of how significant the inaccuracy of the approximation is, consider the lognormal severity distribution F_b with the mean parameter 5 and the standard deviation parameter 3. The comparison of the approximate lower bound $F_b^{-1}(q_1)$ with the true lower bound $F_b^{-1}(q)$ shows that the approximation errors are significant for the smaller values of M, such as 2, 5, and 10. For example, when $M = 2$ and $\lambda = 10$ ($\lambda = 500$), the first order approximation of the lower bound is 64% (34%) higher than the true lower bound. The result is similar even when $M = 5$, ($M = 10$) and $\lambda = 100$, and the first order approximation is 11% (5%) higher than the true lower bound. If the GPD severity is used, the error of the approximation is even higher. In general, the inaccuracy of the approximation (6) increases as the tail of the severity distribution becomes heavier.

5 Alternative approaches to incorporating scenarios

Below, I present five alternative approaches to incorporating scenarios into operational risk modeling. A common feature of all these approaches is that the following inequalities need to be satisfied (see Proposition 4):

$$F^{-1}(q_i) \geq L_i^w, \ i = 1, ..., r, \quad (7)$$

where

$$q_i = 1 + \frac{1}{\lambda} \log\left(1 - \frac{1}{M_i^w}\right), i = 1, ..., r.$$

$F(\cdot) = F_a(\cdot)$ is a scenario-adjusted cdf of the severity distribution for first three approaches, and $F(\cdot) = F_s(\cdot)$ is a scenario-driven cdf for the last two approaches.

The first three approaches allow for the direct incorporation of scenario analysis into the quantification framework through the derivation of a scenario-adjusted cdf, which is then used to calculate scenario-adjusted capital. The first one, the stochastic dominance approach, shifts the cdf of the base model's severity distribution to the right so that all the inequalities of (7) hold. Therefore, this approach should always result in higher scenario-adjusted capital than the base model's capital. The other two approaches, the constraint estimation approach and the constrained Markov chain Monte Carlo (MCMC) approach, incorporate the constraints (7) inside the estimation process.

The last two approaches incorporate the scenario information into the quantification framework indirectly, because under these approaches no need exists to find a scenario-adjusted cdf. Instead, the curve fitting approach leads to a scenario-driven cdf as a curve that is the closest to the set of points $(q_i, L_i^w), i = 1, ..., andr$ under the scenario constraints (7). The fitted cdf curve is then used to calculate the scenario-driven capital, which can be considered as a benchmark for the base model's capital number. Ideally, this curve should make the inequalities (7) binding by turning them into equalities, although this change might not always be possible. Under the minimum distance approach, it is assumed that several competing severity distributions fit the base model reasonably well. Using the minimum distance approach allows me to then choose the severity distribution with the smallest deviation from the scenario constraints (7). Now I turn to a more detailed description of each approach.

The stochastic dominance approach The main idea of this approach is to adjust the cdf of the base model's severity distribution so that the adjusted cdf satisfies the scenario constraints (7). The adjustment is done in such a way that, at any quantile level, the corresponding loss amount (i.e., the inverse of the adjusted severity's cdf at that quantile level) is greater or equal to the loss amount implied by the base model's severity distribution. In this sense, the adjusted cdf stochastically dominates the original cdf (see, e.g., Levy 2006). Thus, the scenario-adjusted capital is always greater than the base model's capital as long as some adjustment is made.

To explain how the adjustment works, I assume that none of the inequalities of (7) hold. In other words, the size of the adjustment is non-negative for all scenarios from $\mathbf{S}^w$:

$$\Delta_i = L_i^w - F_b^{-1}(q_i) \geq 0.$$

If some inequalities of (7) hold, then these inequalities have to be removed before making any adjustments and working with the rest of the inequalities. If all of the inequalities hold, then no need exists to make any adjustments, because all of the scenarios in the set $\mathbf{S}^w$ are concordant with the base model. The adjustments are made piece by piece at the following quantile intervals $(0, q_1], ..., (q_{d-1}, q_r]$, and $(q_r, 1)$. Specifically, the definition of the cdf of the scenario-adjusted severity is:

$$
\begin{aligned}
F_a^{-1}(z) &= F_b^{-1}(z) + \Delta_1 \quad \text{if} \quad z \in (0, q_1] \\
&= F_b^{-1}(z) + \frac{q_i - z}{q_i - q_{i-1}} \Delta_{i-1} + \frac{z - q_{i-1}}{q_i - q_{i-1}} \Delta_i \quad \text{if} \quad z \in (q_{i-1}, q_i], i = 2, ..., r \\
&= F_b^{-1}(z) + \Delta_r \quad \text{if} \quad z \in (q_r, 1)
\end{aligned}
$$

To calculate the regulatory capital, one needs to sample random losses from the severity distribution $F_a(\cdot)$. All one has to do is to sample a random loss, say X, from the base severity distribution $F_b(\cdot)$ and find $z_x = F_b(X)$. Then X is transformed into a random loss, Y, from the scenario-adjusted severity distribution $F_a(\cdot)$ as

$$
\begin{aligned}
Y &= X + \Delta_1 \quad \text{if} \quad z_x \in (0, q_1] \\
&= X + \frac{q_i - z_x}{q_i - q_{i-1}} \Delta_{i-1} + \frac{z_x - q_{i-1}}{q_i - q_{i-1}} \Delta_i \quad \text{if} \quad z_x \in (q_{i-1}, q_i], i = 2, ..., r \\
&= X + \Delta_r \quad \text{if} \quad z_x \in (q_r, 1).
\end{aligned}
$$

Alternatively, one can use the following more conservative and straightforward adjustment: $F_a^{-1}(z) = F_b^{-1}(z) + \Delta^*, \quad z > 0$, where $\Delta^* = \max\{0, \Delta_1, ..., \Delta_r\}$. Sampling from this scenario-adjusted severity distribution is also straightforward: take a random sample from $F_b(\cdot)$ and increase it by Δ^*.

The constrained estimation approach From the classical perspective, (7) indicates a straightforward approach to incorporating scenarios. Specifically, one fits a severity distribution to historical losses by a constrained optimization where the unknown parameter is the estimates under the constraint in which the inequalities (7) hold. As a result, one obtains the parameter estimates of the scenario-adjusted severity distribution.

The constrained MCMC approach The Bayesian perspective incorporates the constraints (7) inside the MCMC algorithm by verifying that the constraints are satisfied every time a parameter or a block of parameters is sampled. As a result, the parameter estimates will automatically satisfy the constraints. I refer the reader to a review by Chib (2001) for an extensive discussion of the MCMC methods.

The curve fitting approach Under this approach, the severity distribution $F_s(\cdot)$ can belong to the same family of distributions as the severity of the base model and can minimize the distance between the right-hand side and the left-hand side of the inequalities (7). The parameters of this cdf curve can be found, for example, by minimizing the following quantile distance:

$$\sum_{i=1}^{r}\left[F_s^{-1}(q_i)-L_i^w\right]^2,$$

under the constraints (7). Once the scenario-driven severity distribution $F_s(\cdot)$ is available, one can calculate the scenario-driven capital as well. Therefore, this approach is appropriate for the purpose of using scenario analysis as a benchmark relative to the base model's capital number.

The minimum distance approach In this approach, the assumption is that several alternative severity distributions lead to a reasonably good fit under the base model. Among these alternative distributions, the choice of the distribution with the right-tail behavior most closely resembling the tail behavior imbedded in the worst-case scenarios is the most likely. Therefore, the preference is to come up with some metric that measures a distance between a severity distribution and the scenarios at the worst-case-scenario lower bounds. Then the choice would be the severity distribution that minimizes that distance. For example, assuming that $J > 1$ severity distributions exist with the cdfs $F_j(\cdot), j = 1, ..., J$ that satisfy the constraints (7), the choice might be the severity distribution with the cdf $F_h(\cdot)$ as the best fit, where $h = argmin_{j=1,...,J}Q_j$ is the argument of the minimum among the quantities:

$$Q_j = \sum_{i=1}^{r}\left[F_j^{-1}(q_i)-L_i^w\right]^2, j = 1, ..., J.$$

It should be noted that removing the scenario constraints relaxes the last two approaches. In this case, these two approaches lead to scenario-driven capital numbers that are not influenced by the built-in conservatism of the proposed framework. However, if the scenarios are systematically understated as a result of known scenario biases so that all scenarios become concordant, then without the built-in conservatism the curve fitting approach results in an understated scenario-driven capital benchmark, and the minimum distance approach might as well lead to the choice of a severity distribution with understated capital. Meanwhile, the scenario-adjusted capital numbers coming out of the first three approaches should be the same as the capital number of the base model when all scenarios are concordant. If expert assessments of the scenarios' lower bounds are systematically overstated so that all scenarios are discordant with the base model, then all approaches should indicate overstated scenario-adjusted or scenario-driven capital numbers. In summary, while all approaches transfer systematic overstatements of the severity of scenario losses to the capital numbers, the built-in conservatism makes it difficult for the scenario-adjusted capital to fall below the base model's capital even when scenario experts substantially understate the severity of scenario losses.

6 Conclusion

In this paper, I propose a theoretically justified framework for incorporating scenario losses into operational risk modeling. The basis of this framework is the idea that one needs to focus on worst-case scenarios, because only those scenarios contain valuable information about the tail behavior of operational losses. I propose a simple rule for identifying the worst-case scenarios from the pool of all scenarios. The information contained in the worst-case scenarios enters the quantification process in the form of lower bound constraints on the specific quantile levels of the severity distribution. Because these quantile levels are quite high even for, e.g., the worst-in-two-year scenarios, I conclude that modelers should avoid the inclusion of scenarios that have a duration of beyond 100 years.

This framework gives rise to at least five approaches for the incorporation of scenarios. The stochastic dominance approach shifts the cdf of the base model's severity distribution to the right to satisfy all scenario constraints. The constraint estimation and the constrained MCMC approaches incorporate the scenario constraints inside the estimation process. The curve fitting approach finds a scenario-driven cdf as a curve that is the closest to the binding scenario constraints. This approach then uses the cdf curve to calculate scenario-driven capital, which the approach considers as a benchmark for the base model's capital number. Under the minimum distance approach, the assumption is that several competing severity distributions exist that fit the base model reasonably well. Using the minimum distance approach one then chooses the severity distribution with the smallest deviation from the binding scenario constraints. Further research is needed to study the practical performance of the proposed approaches.

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